UNRESTRICTED VIRTUAL BRAIDS, FUSED LINKS AND OTHER QUOTIENTS OF VIRTUAL BRAID GROUPS

VALERIY G. BARDAKOV, PAOLO BELLINGERI, AND CELESTE DAMIANI

Abstract. We consider the group of unrestricted virtual braids, describe its structure and explore its relations with fused links. Also, we define the groups of flat virtual braids and virtual Gauss braids and study some of their properties, in particular their linearity.

1. Introduction

Fused links were defined by L. H. Kauffman and S. Lambropoulou in [22]. Afterwards, the same authors introduced their "braided" counterpart, the *unrestricted virtual braids*, and extended S. Kamada's work ([18]) by presenting a version of Alexander and Markov theorems for these objects [23]. In the *group of unrestricted virtual braids*, which will be denoted by UVB_n, we consider braid-like diagrams in which we allow two kinds of crossing (classical and virtual), and where the equivalence relation is given by ambient isotopy and by the following transformations: classical Reidemeister moves (Figure 1), virtual Reidemeister moves (Figure 2), a mixed Reidemeister move (Figure 3), and two moves of type Reidemeister III with two classical crossings and one virtual crossing (Figure 4). These two last moves are called *forbidden moves.*

The group UVB_n appears also in [17], where it is called *symmetric loop braid group*, being a quotient of the *loop braid group* LB_n studied in [2], also known as the welded braid group WB_n.

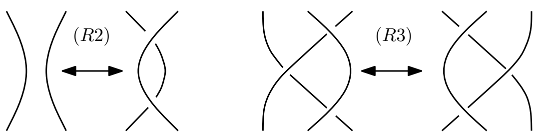

Figure 1. Classical Reidemeister moves.

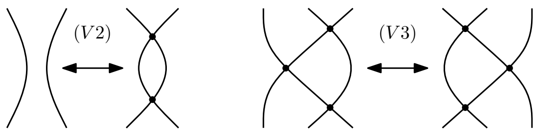

Figure 2. Virtual Reidemeister moves.

1991 *Mathematics Subject Classification.* Primary 20F36.

Key words and phrases. Braid groups, virtual and welded braids, virtual and welded knots, group of knot.

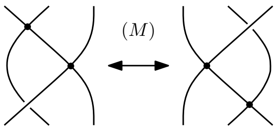

FIGURE 3. Mixed Reidemeister move.

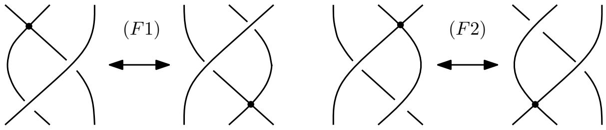

FIGURE 4. Forbidden moves of type (F1) (on the left) and type (F2) (on the right).

It has been shown that all fused knots are equivalent to the unknot ([19,28]). Moreover, S. Nelson's proof in [28] of the fact that every virtual knot unknots, when allowing forbidden moves, which is carried on using Gauss diagrams, can be adapted verbatim to links with several components. So, every fused link diagram is fused isotopic to a link diagram where the only crossings (classical or virtual) are the ones involving different components.

On the other hand, there are non-trivial fused links and their classification is not (completely) trivial ([13]): in particular in [12], A. Fish and E. Keyman proved that fused links that have only classical crossings are characterized by their (classical) linking numbers. However, this result does not generalize to links with virtual crossings: in fact it is easy to find non-equivalent fused links with the same (classical) linking number (see Remark 3.8). This answers a question from [12, Remark 1], where Fish and Keyman ask whether the classical linking number is a complete invariant for fused links.

The first aim of this note is to give a short survey on above knotted objects, describe unrestricted virtual braids and compare more or less known invariants for fused links. In Section 2 we give a description of the structure of the group of unrestricted virtual braids UVB_n (Theorems 2.4 and 2.7), answering a question of Kauffman and Lambropoulou from [23]. In Section 3 we provide an application of Theorem 2.7 showing that any fused link admits as a representative the closure of a *pure* unrestricted virtual braid (Theorem 3.6); as a corollary we deduce an easy proof of the theorem of Fish and Keyman cited in previous paragraph. In Section 4 we construct a representation for UVB_n in $\mathrm{Aut}(N_n)$, the group of automorphisms of the free 2-step nilpotent group of rank n (Proposition 4.5). Using this representation we define a notion of group of fused links and we compare this invariant to other known invariants (Proposition 4.10 and Remark 4.11).

Finally, in Section 5 we describe the structure of other quotients of virtual braid groups: the flat virtual braid group (Proposition 5.1 and Theorem 5.3), the flat welded braid group (Proposition 5.5) and the virtual Gauss braid group (Theorem 5.7). As a corollary we prove that flat virtual braid groups and virtual Gauss braid groups are linear and that they have solvable word problem (the fact that unrestricted virtual braid groups are linear and have solvable word problem is a trivial consequence of Theorem 2.7).

Acknowledgments. The research of the first author was partially supported by by Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020), RFBR-14-01-00014, RFBR-15-01-00745 and Indo-Russian RFBR-13-01-92697. The research of the second author was partially supported by French grant ANR-11-JS01-002-01. This paper was started when the first author was in Caen. He thanks the members of the Laboratory of Mathematics of the University of Caen for their invitation and hospitality. The authors are deeply indebted to the anonymous referee for valuable remarks and suggestions.

2. Unrestricted virtual braid groups

In this Section, in order to define unrestricted virtual braid groups, we will first introduce virtual and welded braid groups by simply recalling their group presentation; for other definitions, more intrinsic, see for instance [3, 10, 18, 31] for the virtual case and [9, 11, 18] for the welded one.

Definition 2.1. The *virtual braid group* VB_n is the group defined by the group presentation

$$\langle \{\sigma_i, \rho_i \mid i = 1, \dots, n-1\} \mid R \rangle$$

where R is the set of relations:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $i = 1, \dots, n-2$;

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i-j| \geq 2$;

(R3) $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$, for $i = 1, \dots, n-2$;

(R4) $\rho_i \rho_j = \rho_j \rho_i$, for $|i-j| \geq 2$;

(R5) $\rho_i^2 = 1$, for $i = 1, \dots, n-1$;

(R6) $\sigma_i \rho_j = \rho_j \sigma_i$, for $|i-j| \geq 2$;

(M) $\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}$, for $i = 1, \dots, n-2$.

We define the *virtual pure braid group*, denoted by VP_n, to be the kernel of the map $VB_n \to S_n$ sending, for every $i = 1, 2, \dots, n-1$, generators σ_i and ρ_i to $(i, i+1)$. A presentation for VP_n is given in [4]; it will be recalled in the proof of Theorem 2.7 and Proposition 5.1.

The welded braid group WB_n can be defined as a quotient of VB_n by the normal subgroup generated by relations

(F1) $\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}$, for $i = 1, \dots, n-2$.

Remark 2.2. We will see in Section 3 that the symmetrical relations

(F2) $\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i$, for $i = 1, \dots, n-2$

do not hold in WB_n. This justifies Definition 2.3.

Definition 2.3. We define the *group of unrestricted virtual braids* UVB_n as the group defined by the group presentation

$$\langle \{\sigma_i, \rho_i \mid i = 1, \dots, n-1\} \mid R' \rangle$$

where R' is the set of relations (R1), (R2), (R3), (R4), (R5), (R6), (M), (F1), (F2).

The main result of this Section is to prove that UVB_n can be described as semi-direct product of a right-angled Artin group and the symmetric group S_n: this way we answer a question posed in [23] about the (non-trivial) structure of UVB_n.

Theorem 2.4. *Let X_n be the right-angled Artin group generated by $x_{i,j}$ for $1 \leq i \neq j \leq n$ where all generators commute except the pairs $x_{i,j}$ and $x_{j,i}$ for $1 \leq i \neq j \leq n$. The group UVB_n is isomorphic to $X_n \rtimes S_n$ where S_n acts by permutation on the indices of generators of X_n.*

Let $\nu\colon UVB_n \to S_n$ be the map defined as follows:

$$\nu(\sigma_i) = \nu(\rho_i) = (i, i+1), \quad \text{for } i = 1, 2, \ldots, n-1.$$

We will call the kernel of ν *unrestricted virtual pure braid group* and we will denote it by UVP_n. Since ν admits a natural section, we have that $UVB_n = UVP_n \rtimes S_n$.

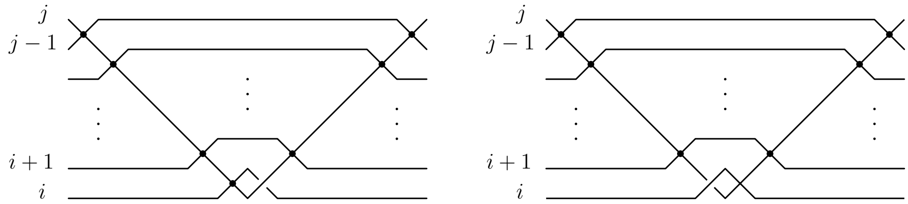

FIGURE 5. Elements $\lambda_{i,j}$ on the right and $\lambda_{j,i}$ on the left. Here we adopt the convention of drawing braids from left to right.

Let us define some elements of UVP_n (see Figure 5). For $i = 1, \ldots, n-1$:

$$\begin{aligned} \lambda_{i,i+1} &= \rho_i \sigma_i^{-1}, \\ \lambda_{i+1,i} &= \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i. \end{aligned} \tag{1}$$

For $1 \leq i < j-1 \leq n-1$:

$$\begin{aligned} \lambda_{i,j} &= \rho_{j-1}\rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2}\rho_{j-1}, \\ \lambda_{j,i} &= \rho_{j-1}\rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2}\rho_{j-1}. \end{aligned} \tag{2}$$

The next lemma was proved in [4] for the corresponding elements in VB_n and therefore is also true in the quotient UVB_n.

Lemma 2.5. *The following conjugating rule is fulfilled in UVB_n: for all $1 \leq i \neq j \leq n$ and $s \in S_n$,*

$$\iota(s) \lambda_{i,j} \iota(s)^{-1} = \lambda_{s(i),s(j)}$$

where $\iota\colon S_n \to UVB_n$ is the natural section of the map ν defined in Theorem 2.4.

Corollary 2.6. *The group S_n acts by conjugation on the set $\{\lambda_{k,l} \mid 1 \leq k \neq l \leq n\}$. This action is transitive.*

We prove that the group generated by $\{\lambda_{k,l} \mid 1 \leq k \neq l \leq n\}$ coincides with UVP_n, and then we will find the defining relations. This will show that UVP_n is a right-angled Artin group.

Theorem 2.7. *The group UVP_n admits a presentation with generators $\lambda_{k,l}$ for $1 \leq k \neq l \leq n$, and defining relations: $\lambda_{i,j}$ commute with $\lambda_{k,l}$ if and only if $k \neq j$ or $l \neq i$.*

Proof. Since UVP_n is a finite index subgroup of UVB_n one can apply Reidemeister–Schreier method (see, for example, [25, Ch. 2.2]) and check that the given set of relations is complete. Remark that most of the relations were already proven in this way in [4] for the case of the virtual pure braid group VP_n.

An easier approach is provided by the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker \pi_{|VP_n} & \longrightarrow & \ker \pi & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & VP_n & \longrightarrow & VB_n & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_{|VP_n}} & & \downarrow{\scriptstyle \pi} & & \| & & \\
1 & \longrightarrow & UVP_n & \longrightarrow & UVB_n & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where π is the canonical projection of VB_n onto UVB_n and $\pi_{|VP_n}$ its restriction to VP_n. By definition $\ker \pi$ is normally generated by elements $\sigma_i \sigma_j \rho_i \sigma_j^{-1} \sigma_i^{-1} \rho_j$ for $|i-j| = 1$ (we will write $\ker \pi = \ll \sigma_i \sigma_j \rho_i \sigma_j^{-1} \sigma_i^{-1} \rho_j |$ for $|i-j| = 1 \gg$). Since $\sigma_i \sigma_j \rho_i \sigma_j^{-1} \sigma_i^{-1} \rho_j$ belongs to VP_n and that VP_n is normal in VB_n, we deduce that $\ker \pi_{VP_n}$ coincides with $\ker \pi$.

We recall that, according to [4], VP_n is generated by elements $\lambda_{i,j}$ defined as follows:

$$\begin{aligned}
\lambda_{i,i+1} &= \rho_i \sigma_i^{-1}, \\
\lambda_{i+1,i} &= \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i .
\end{aligned} \tag{3}$$

For $1 \leq i < j-1 \leq n-1$:

$$\begin{aligned}
\lambda_{i,j} &= \rho_{j-1}\rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2}\rho_{j-1}, \\
\lambda_{j,i} &= \rho_{j-1}\rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2}\rho_{j-1}.
\end{aligned} \tag{4}$$

and has the following set of defining relations:

$$\lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j} \tag{RS1}$$

$$\lambda_{k,i}(\lambda_{k,j}\lambda_{i,j}) = (\lambda_{i,j}\lambda_{k,j})\lambda_{k,i}. \tag{RS2}$$

Moreover, as UVB_n, VB_n can be seen as a semidirect product $VP_n \rtimes S_n$, where the symmetric group S_n acts by permutations of indices on $\lambda_{i,j}$'s (Lemma 2.5).

One can easily verify that relators of type (F1), *i.e.*, $\rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}$, can be rewritten as:

$$(\rho_i \, \lambda_{i+1,i+2}^{-1} \, \rho_i)(\rho_i \, \rho_{i+1} \, \lambda_{i,i+1}^{-1} \, \rho_{i+1} \, \rho_i)(\rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1})\lambda_{i+1,i+2}$$

and using the conjugating rule given above, we get, for $i = 1, \ldots, n-2$,

$$\rho_i\, \sigma_{i+1}\sigma_i\, \rho_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \lambda_{i,i+2}^{-1}\, \lambda_{i+1,i+2}^{-1}\, \lambda_{i,i+2}\, \lambda_{i+1,i+2}.$$

On the other hand one can similarly check that relators of type (F2), which are of the form $\rho_{i+1}\sigma_i\, \sigma_{i+1}\rho_i\, \sigma_{i+1}^{-1}\sigma_i^{-1}$, can be rewritten as $\lambda_{i,i+1}^{-1}\lambda_{i,i+2}^{-1}\lambda_{i,i+1}\lambda_{i,i+2}$.

From this facts and from above description of VB_n as semidirect product $VP_n \rtimes S_n$, it follows that any generator of $\ker_{VP_n}$ is of the form $g[\lambda_{i,j}, \lambda_{k,j}]g^{-1}$ or $h[\lambda_{i,j}, \lambda_{i,k}]h^{-1}$ for $g, h \in VP_n$ and i, j, k distincts. The group UVP_n has therefore the following complete set of relations

$$\lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j} \tag{RS1}$$

$$\lambda_{k,i}(\lambda_{k,j}\lambda_{i,j}) = (\lambda_{i,j}\lambda_{k,j})\lambda_{k,i} \tag{RS2}$$

$$\lambda_{i,j}\lambda_{k,j} = \lambda_{k,j}\lambda_{i,j} \tag{RS3}$$

$$\lambda_{i,j}\lambda_{i,k} = \lambda_{i,k}\lambda_{i,j}. \tag{RS4}$$

Using (RS3) and (RS4) we can rewrite relation (RS2) in the form

$$\lambda_{k,j}(\lambda_{k,i}\lambda_{i,j}) = \lambda_{k,j}(\lambda_{i,j}\lambda_{k,i}). \tag{5}$$

After cancelation we have that we can replace relation (RS2) with

$$\lambda_{k,i}\lambda_{i,j} = \lambda_{i,j}\lambda_{k,i} \tag{RS5}$$

This completes the proof. □

Proof of Theorem 2.4. The group X_n is evidently isomorphic to UVP_n (sending any $x_{i,j}$ into the corresponding $\lambda_{i,j}$). Recall that UVP_n is the kernel of the map $\nu\colon UVB_n \to S_n$ defined as $\nu(\sigma_i) = \nu(\rho_i) = (i, i+1)$ for $i = 1, \ldots, n-1$. Recall also that ν has a natural section $\iota\colon S_n \to UVB_n$, defined as $\iota((i, i+1)) = \rho_i$ for $i = 1, \ldots, n-1$. Therefore UVB_n is isomorphic to $UVP_n \rtimes S_n$ where S_n acts by permutation on the indices of generators of UVP_n (see Corollary 2.6). □

We recall that the pure braid group P_n is the kernel of the homomorphism from B_n to the symmetric group S_n sending every generator σ_i to the permutation $(i, i+1)$. It is generated by the set $\{a_{ij} \mid 1 \le i < j \le n\}$, where

$$a_{i,i+1} = \sigma_i^2,$$
$$a_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \quad \text{for } i+1 < j \le n.$$

Corollary 2.8. *Let $p\colon P_n \to UVP_n$ be the canonical map of the pure braid group P_n in UVP_n. Then $p(P_n)$ is isomorphic to the abelianization of P_n.*

Proof. As remarked in ([4, page 6]), generators $a_{i,j}$ of P_n can be rewritten in VP_n as

$$a_{i,i+1} = \lambda_{i,i+1}^{-1}\lambda_{i+1,i}^{-1}, \quad \text{for } i = 1, \ldots, n-1,$$
$$a_{i,j} = \lambda_{j-1,j}^{-1}\lambda_{j-2,j}^{-1}\cdots\lambda_{i+1,j}^{-1}(\lambda_{i,j}^{-1}\lambda_{j,i}^{-1})\lambda_{i+1,j}\cdots\lambda_{j-2,j}\lambda_{j-1,j}, \quad \text{for } 2 \le i+1 < j \le n,$$

and therefore in UVP_n we have:

$$p(a_{i,i+1}) = \lambda_{i,i+1}^{-1}\lambda_{i+1,i}^{-1}, \quad \text{for } i = 1, \dots, n-1,$$
$$p(a_{i,j}) = \lambda_{j-1,j}^{-1}\lambda_{j-2,j}^{-1}\cdots\lambda_{i+1,j}^{-1}(\lambda_{i,j}^{-1}\lambda_{j,i}^{-1})\lambda_{i+1,j}\cdots\lambda_{j-2,j}\lambda_{j-1,j}, \quad \text{for } 2 \leq i+1 < j \leq n.$$

According to Theorem 2.7, UVP_n is the cartesian product of the free groups of rank 2 $F_{i,j} = \langle \lambda_{i,j}, \lambda_{j,i} \rangle$ for $1 \leq i < j \leq n$.

For every generator $a_{i,j}$ for $1 \leq i < j \leq n$ of P_n we have that its image is in $F_{i,j}$ and it is not trivial. In fact, $p(a_{i,j}) = \lambda_{i,j}^{-1}\lambda_{j,i}^{-1}$. So $p(P_n)$ is isomorphic to $\mathbb{Z}^{n(n-1)/2}$. The statement therefore follows readily since the abelianized of P_n is $\mathbb{Z}^{n(n-1)/2}$. □

3. Unrestricted virtual braids and fused links

Definition 3.1. A *virtual link diagram* is a closed oriented 1-manifold D immersed in $\mathbb{R}^2$ such that all multiple points are transverse double points, and each double point is provided with an information of being *positive*, *negative* or *virtual* as in Figure 6. We assume that virtual link diagrams are the same if they are isotopic in $\mathbb{R}^2$. Positive and negative crossings will also be called *classical crossings*.

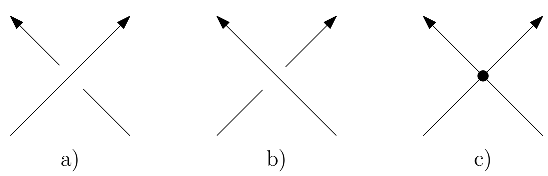

FIGURE 6. a) Positive crossing, b) Negative crossing, c) Virtual crossing.

Definition 3.2. *Fused isotopy* is the equivalence relation on the set of virtual link diagrams given by classical Reidemeister moves, virtual Reidemeister moves, the mixed Reidemeister move (M), and the forbidden moves (F1) and (F2).

Remark 3.3. These moves are the moves pictured in Figure 1, 2, 3, and 4, with the addition of Reidemeister moves of type I, both classical and virtual, see Figure 7.

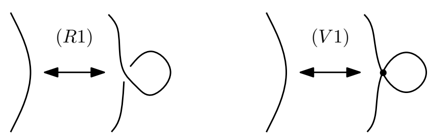

FIGURE 7. Reidemeister moves of type I.

Definition 3.4. A *fused link* is an equivalence class of virtual link diagrams with respect to fused isotopy.

The classical Alexander Theorem generalizes to virtual braids and links, and it directly implies that every oriented welded (resp. fused) link can be represented by a welded (resp. unrestricted virtual) braid, whose Alexander closure is isotopic to the original link. Two braiding algorithms are given in [18] and [22].

Similarly we have a version of Markov Theorem ([23]): before stating it, we recall that the natural map $UVB_n \to UVB_{n+1}$, that adds one strand on the right of an element of UVB_n, with the convention of considering braids going from the top to the bottom, is an inclusion.

Theorem 3.5 ([23])**.** *Two oriented fused links are isotopic if and only if any two corresponding unrestricted virtual braids differ by moves defined by braid relations in UVB_∞ (braid moves) and a finite sequence of the following moves (extended Markov moves):*

- *Virtual and classical conjugation: $\rho_i \beta \rho_i \sim \beta \sim \sigma_i^{-1} \beta \sigma_i \sim \sigma_i \beta \sigma_i^{-1}$;*
- *Right virtual and classical stabilization: $\beta \rho_n \sim \beta \sim \beta \sigma_n^{\pm 1}$;*

where $UVB_\infty = \bigcup_{n=2}^{\infty} UVB_n$, β is a braid in UVB_n, σ_i, ρ_i generators of UVB_n and $\sigma_n, \rho_n \in UVB_{n+1}$.

Here we give an application to fused links of Theorem 2.4.

Theorem 3.6. *Any fused link is fused isotopic to the closure of an unrestricted virtual pure braid.*

Proof. Let us start remarking that the case of knots is trivial because knots are fused isotopic to the unknot ([19, 28]).

Let now L be a fused link with $n > 1$ components; then there is an unrestricted virtual braid $\alpha \in UVB_m$ such that $\hat{\alpha}$ is fused isotopic to L.

Let $s_{kl} = \rho_{k-1} \rho_{k-2} \cdots \rho_l$ for $l < k$ and $s_{kl} = 1$ in other cases. We define the set

$$\Lambda_n = \left\{ \prod_{k=2}^{n} s_{k,j_k} | 1 \leq j_k \leq k \right\}$$

which can be seen as the "virtual part" of UVB_n, since it coincides with the set of canonical forms of elements in $\iota(S_n)$, where ι is the map from Lemma 2.5.

Then using Theorem 2.4 we can rewrite α as:

$$\alpha = l_{1,2} l_{1,3} l_{2,3} \cdots l_{m-1,m} \pi$$

where $l_{i,j} \in \langle \lambda_{i,j}, \lambda_{j,i} \rangle$ and $\pi = s_{2,j_2} \cdots s_{m,j_m} \in \Lambda_n$ (see Figure 8).

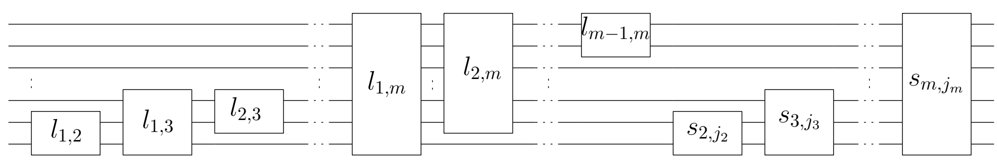

FIGURE 8. The braid α.

Using Lemma 2.5, we can do another rewriting:

$$\alpha = L_2 s_{2,j_2} L_3 s_{3,j_3} \cdots L_m s_{m,j_m}$$

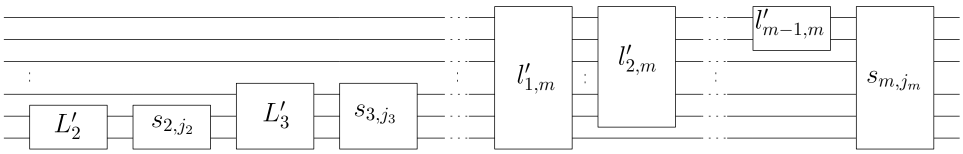

FIGURE 9. The rewriting of the braid α, with $L'_i \in \langle \lambda_{1,i}, \lambda_{i,1} \rangle \times \cdots \times \langle \lambda_{i-1,i}, \lambda_{i,i-1} \rangle$.

where $L_i \in \langle \lambda_{1,i}, \lambda_{i,1} \rangle \times \cdots \times \langle \lambda_{i-1,i}, \lambda_{i,i-1} \rangle$.

Then again we can reorder terms in the L_is:

$$\alpha = l'_{1,2} s_{2,j_2} l'_{1,3} l'_{2,3} s_{3,j_3} \cdots l'_{m-1,m} s_{m,j_m}$$

with $l'_{i,j} \in \langle \lambda_{i,j}, \lambda_{j,i} \rangle$, and (see Figure 9).

If $s_{i,j_i} = 1$ for $i = 2, \ldots, m$, then α is a pure braid and $m = n$.

Suppose then that there is a $s_{k,j_k} \neq 1$ for some k, and that $s_{i,j_i} = 1$ for each $i > k$.

Conjugating α for $s_{m,1}^{m-k}$, we obtain a braid $\alpha_1 = s_{m,1}^{k-m} \alpha s_{m,1}^{m-k}$ whose closure is fused isotopic to L where the k-th strand of α is the m-th strand of α_1.

We can rewrite α_1 as:

$$\alpha_1 = \gamma \; l''_{1,m} l''_{2,m} \cdots l''_{m-1,m} \; s_{m,k_m}$$

where $\gamma = l'' s_{2,j_2} \cdots l''_{m-2,m-1} s_{m-1,k_{m-1}}$, so it does not involve the m-th strand, and $l''_{1,m} l''_{2,m} \cdots l''_{m-1,m}$ is pure. For definition $s_{m,k_m} = \rho_{m-1} s_{m-1,k_m}$. The m-th strand and the other strand involved in this occurrence of ρ_{m-1} that we have just isolated, belong to the same component of $L_1 = \hat{\alpha_1}$ (see Figure 10). Hence also all the crossings in $l''_{m-1,m}$ belong to that same component.

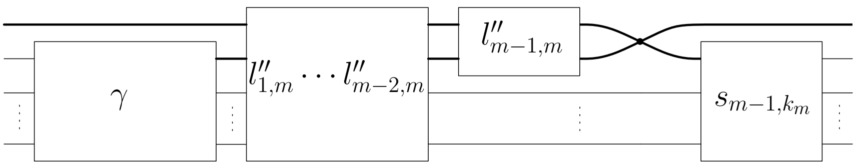

FIGURE 10. The form of α_1.

We virtualize all classical crossings of $l''_{m-1,m}$ using Kanenobu's technique ([19, Proof of Theorem 1]): it consists in deforming the understrand of one classical crossing at a time, considered in the closure of the link, with a sequence of generalized Reidemeister moves, pushing it along the whole component. At the end of the process, there is a new classical crossing instead of the original one, and $2j$ new virtual crossings, where j is the number of crossings the understrand has been pushed through. With generalized Reidemeister moves of braid type, one can change the original classical crossing with a virtual one and remove the new classical crossing with a Reidemeister move of type I. Since our crossings are on the top strand, this Reidemeister move of type I is equivalent to a Markov's classical stabilisation, so we obtain a new link L'_1, fused isotopic to L, associated to a braid α'_1 who is identical to α_1 except that it has a virtual crossing at the place of the classical crossing considered. This is done for each classical crossing in $l''_{m-1,m}$.

Since $l''_{m-1,m}$ has an even total number of generators σ_{m-1} and ρ_{m-1}, after virtualizing $l''_{m-1,m}\rho_{m-1}$ becomes a word composed by an odd number of ρ_{m-1}. Applying the relation associated with the virtual Reidemeister move of type II we obtain a new link L_2, fused isotopic to L, associated to $\alpha_2 = \gamma\ l''_{1,m} l''_{2,m} \cdots l''_{m-2,m}\ \rho_{m-1}\ s_{m-1,k_m}$.

Applying once more Lemma 2.5, α_2 becomes $\gamma\rho_{m-1}\ \overline{l_{1,m}}\ \overline{l_{2,m}} \ldots \overline{l_{m-2,m}}\ s_{m-1,k_m}$, where $\overline{l_{i,m}}$ is a word in $\langle \lambda_{m-1,i}, \lambda_{i,m-1} \rangle$.

In α_2 there is only one (virtual) crossing on the m-th strand, so, using Markov moves (conjugation and virtual stabilisation) we obtain a new braid α_3, whose closure is again fused isotopic to L and has $(m-1)$ strands. In other words, the braid α_3 is obtained removing from α_2 the only virtual crossing on the m-th strand, and thanks to Markov theorem its closure is fused isotopic to L.

If we continue this process, eventually we will get to a braid β in UVB_n whose closure is fused isotopic to L. At this point, each strand of β corresponds to a different component of L, so β must be a pure braid. □

The technique used in Theorem 3.6 was used, associated with braid decomposition in B_n, by A. Fish and E. Keyman to prove the following result about fused links.

Theorem 3.7 ([12])**.** *A fused link with only classical crossings L with n components is completely determined by the linking numbers of each pair of components under fused isotopy.*

The proof in [12] is quite technical, it involves several computations on generators of the pure braid group and their images in UVP_n. Previous result allows us to give an easier proof: the advantage is that no preliminary lemma on the properties of the pure braid group generators is necessary.

Proof (of Theorem 3.7). We consider a fused link with only classical crossings L with n components: when applying Kanenobu's technique to obtain α_2 (see the proof of Theorem 3.6), one gets a braid with only one virtual crossing on the m-strand, and removes it, so that the resulting braid α_3 only has classical crossings.

So, continuing the process, one gets that L is fused isotopic to the closure of an n-string unrestricted virtual pure braid β which only has classical crossings.

Even though B_m and P_m are not subgroups of UVB_m, since $\hat{\beta}$ has only classical crossings, we can consider B_m and P_m's images in UVB_m and rewrite the pure braid β in terms of $a_{i,j}$ generators, and conclude as Fish and Keyman do, defining a group homomorphism $\delta_{i,j} : P_n \to \mathbb{Z}$ by

$$a_{s,t} \mapsto \begin{cases} 1 & \text{if } s = i \text{ and } t = j; \\ 0 & \text{otherwise} \end{cases}$$

which is the classical linking number $lk_{i,j}$ of L's i-th and j-th components. Any fused link with only classical crossings L with n components can be obtained as a closure of a pure braid $\beta = x_2 \cdots x_n$ where each x_i can be written in the form $x_i = a_{1,i}^{\delta_{1,i}} \cdots a_{i-1,i}^{\delta_{i-1,i}}$ (Corollary 2.8). This shows that β only depends on the linking number of the components. □

In [15, Section 1] a *virtual* version of the *linking number* is defined in the following way: to a 2-component link we associate a couple of integers $(vlk_{1,2}, vlk_{2,1})$ where $vlk_{1,2}$ is the sum of signs of classical crossings where the first component passes over the second one, while $vlk_{2,1}$ is computed by exchanging the components in the definition of $vlk_{1,2}$. Clearly the classical linking number $lk_{1,2}$ is equal to half the sum of $vlk_{1,2}$ and $vlk_{2,1}$.

Using this definition of virtual linking number, we could be tempted to extend Fish and Keyman results, claiming that a fused link L is completely determined by the virtual linking numbers of each pair of components under fused isotopy.

However for the unrestricted case the previous argument cannot be straightforwardly applied: the virtual linking number is able to distinguish $\lambda_{i,j}$ from $\lambda_{j,i}$, but it is still an application from UVP_n to $(\mathbb{Z}^2)^{n(n-1)/2} = \mathbb{Z}^{n(n-1)}$ that counts the exponents (*i.e.*, the number of appearances) of each generator. Since UVP_n isn't abelian, this is not sufficient to completely determine the braid.

Remark 3.8. Fish and Keynman in [12] suggest that their theorem cannot be extended to links with virtual crossings between different components. They consider the unlink on two components U_2 and $L = \hat{\alpha}$, where $\alpha = \sigma_1 \rho_1 \sigma_1^{-1} \rho_1$, they remark that their classical linking number is 0 but they conjecture that these two links are not fused isotopic. In fact, considering the virtual linking number we can see that $(vlk_{1,2}, vlk_{2,1})(U_2) = (0, 0)$, while $(vlk_{1,2}, vlk_{2,1})(L) = (-1, 1)$.

4. The fused link group

4.1. **A representation for the unrestricted virtual braid group.** Let us recall that the braid group B_n may be represented as a subgroup of $\mathrm{Aut}(F_n)$ by associating to any generator σ_i, for $i = 1, 2, \ldots, n-1$, of B_n the following automorphism of F_n:

$$\sigma_i : \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases} \tag{6}$$

Moreover Artin provided (see for instance [16, Theorem 5.1]) a characterization of braids as automorphisms of free groups: an automorphism $\beta \in \mathrm{Aut}(F_n)$ lies in B_n if and only if β satisfies the following conditions:

i) $\beta(x_i) = a_i\, x_{\pi(i)}\, a_i^{-1}, \quad 1 \leq i \leq n$;
ii) $\beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n$,

where $\pi \in S_n$ and $a_i \in F_n$.

According to [11] we call *group of automorphisms of permutation conjugacy type*, denoted by PC_n, the group of automorphisms satisfying the first condition. The group PC_n is isomorphic to WB_n [11]; more precisely to each generator σ_i of WB_n we associate the previous automorphisms of F_n while to each generator ρ_i, for $i = 1, 2, \ldots, n-1$, we associate the following automorphism of F_n:

$$\rho_i : \begin{cases} x_i \longmapsto x_{i+1} \\ x_{i+1} \longmapsto x_i, \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases} \tag{7}$$

We have thus a faithful representation $\psi\colon WB_n \to \mathrm{Aut}(F_n)$.

Remark 4.1. The group PC_n admits also other equivalent definitions in terms of mapping classes and configuration spaces: it appears often in the literature with different names and notations, such as group of flying rings [3, 9], McCool group [7], motions group [14] and loop braid group [2].

Remark 4.2. Kamada remarks in [18] that the classical braid group B_n embeds in VB_n through the canonical epimorpism $VB_n \to WB_n$. It can be seen via an argument in [11] that B_n is isomorphic to the subgroup of VB_n generated by $\{\sigma_1, \dots, \sigma_n\}$.

Remark 4.3. As a consequence of the isomorphism between WB_n and PC_n, we can show that relation (F2) does not hold in WB_n. In fact applying $\rho_{i+1}\sigma_i\,\sigma_{i+1}$ one gets

$$\rho_{i+1}\sigma_i\,\sigma_{i+1} : \begin{cases} x_i \longmapsto x_i \longmapsto x_i x_{i+1} x_i^{-1} \longmapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1}, \\ x_{i+1} \longmapsto x_{i+2} \longmapsto x_{i+2} \longmapsto x_{i+1}, \\ x_{i+2} \longmapsto x_{i+1} \longmapsto x_i \longmapsto x_i, \end{cases}$$

while applying $\sigma_i\,\sigma_{i+1}\rho_i$ one gets

$$\sigma_i\,\sigma_{i+1}\rho_i : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1} \longmapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \longmapsto x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}, \\ x_{i+1} \longmapsto x_i \longmapsto x_i \longmapsto x_{i+1}, \\ x_{i+2} \longmapsto x_{i+2} \longmapsto x_{i+1} \longmapsto x_i. \end{cases}$$

Since $x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \neq x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}$ in F_n we deduce that relation (F2) does not hold in WB_n.

Our aim is to find a representation for unrestricted virtual braids as automorphisms of a group G. Since the map $\psi\colon WB_n \to \mathrm{Aut}(F_n)$ does not factor through the quotient UVB_n (Remark 4.3) we need to find a representation in the group of automorphisms of a quotient of F_n in which relation (F2) is preserved.

Remark 4.4. In [17] the authors look for representations of the braid group B_n that can be extended to the loop braid group WB_n but do not factor through UVB_n, which is its quotient by relations of type (F2), while we look for a representation that does factor.

Let $F_n = \gamma_1 F_n \supseteq \gamma_2 F_n \supseteq \cdots$ be the lower central series of F_n, the free group of rank n, where $\gamma_{i+1}F_n = [F_n, \gamma_i F_n]$. Let us consider its third term, $\gamma_3 F_n = \big[F_n, [F_n, F_n]\big]$; the free 2-step nilpotent group N_n of rank n is defined to be the quotient ${F_n}/{\gamma_3 F_n}$.

There is an epimorphism from F_n to N_n that induces an epimorphism from $\mathrm{Aut}(F_n)$ to $\mathrm{Aut}(N_n)$ (see [1]). Then, let $\phi\colon UVB_n \to \mathrm{Aut}(N_n)$ be the composition of $\varphi\colon UVB_n \to \mathrm{Aut}(F_n)$ and $\mathrm{Aut}(F_n) \to \mathrm{Aut}(N_n)$.

Proposition 4.5. *The map $\phi\colon UVB_n \to \mathrm{Aut}(N_n)$ is a representation for UVB_n.*

Proof. We use the convention $[x, y] = x^{-1}y^{-1}xy$. In N_n we have that $\big[[x_i, x_{i+1}], x_{i+2}\big] = 1$, for $i = 1, \dots, n-2$, meaning that $x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} = x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}$, *i.e.*, relation (F2) is preserved. $\square$

Proposition 4.6. *The image of the representation $\phi\colon UVP_n \to \mathrm{Aut}(N_n)$ is a free abelian group of rank $n(n-1)$.*

Proof. From Theorem 2.4 we have that the only generators that do not commute in UVP_n are $\lambda_{i,j}$ and $\lambda_{j,i}$ with $1 \leq i \neq j \leq n$.

Recalling the expressions of $\lambda_{i,j}$ and $\lambda_{j,i}$ in terms of generators σ_i and ρ_i, we see that the automorphisms associated to $\lambda_{i,j}$ and $\lambda_{j,i}$ are

$$\phi(\lambda_{i,j}) : \begin{cases} x_i \longmapsto x_j^{-1} x_i x_j = x_i[x_i, x_j] \\ x_k \longmapsto x_k, \text{ for } k \neq i; \end{cases}$$

$$\phi(\lambda_{j,i}) : \begin{cases} x_j \longmapsto x_i^{-1} x_j x_i = x_j[x_j, x_i] = x_j[x_i, x_j]^{-1}; \\ x_k \longmapsto x_k \text{ for } k \neq i. \end{cases}$$

It is then easy to check that the automorphisms associated to $\lambda_{i,j}\lambda_{j,i}$ and to $\lambda_{j,i}\lambda_{i,j}$ coincide:

$$\phi(\lambda_{i,j}\lambda_{j,i}) = \phi(\lambda_{j,i}\lambda_{i,j}) : \begin{cases} x_i \longmapsto x_i[x_i, x_j] \\ x_j \longmapsto x_j[x_i, x_j]^{-1}. \end{cases}$$

To see that in $\phi(UVP_n)$ there is no torsion, let us consider a generic element w of UVP_n. It will have the form $w = l_{1,2} l_{1,3} \cdots l_{n-1,n}$ where $l_{i,j}$ is a product of generators $\lambda_{i,j}$ and $\lambda_{j,i}$.

Generalizing the calculation done above, we have that

$$\phi(l_{1,2} l_{1,3} \cdots l_{n,n-1}) = \phi(\lambda_{1,2}^{\varepsilon_{1,2}} \lambda_{2,1}^{\varepsilon_{2,1}} \cdots \lambda_{n-1,n}^{\varepsilon_{n-1,n}} \lambda_{n,n-1}^{\varepsilon_{n,n-1}}),$$

where $\varepsilon_{i,j}$ is the total number of appearances of $\lambda_{i,j}$ in $l_{i,j}$.

With another easy calculation (check out also Remark 4.11) we have that:

$$\phi(\lambda_{1,2}^{\varepsilon_{1,2}} \lambda_{2,1}^{\varepsilon_{2,1}} \cdots \lambda_{n-1,n}^{\varepsilon_{n-1,n}} \lambda_{n,n-1}^{\varepsilon_{n,n-1}}) : \begin{cases} x_1 \longmapsto x_1[x_1, x_2]^{\varepsilon_{12}}[x_1, x_3]^{\varepsilon_{13}} \cdots [x_1, x_n]^{\varepsilon_{1n}} \\ x_2 \longmapsto x_2[x_2, x_1]^{\varepsilon_{21}}[x_2, x_3]^{\varepsilon_{23}} \cdots [x_2, x_n]^{\varepsilon_{2n}} \\ \vdots \\ x_n \longmapsto x_n[x_n, x_1]^{\varepsilon_{n1}}[x_n, x_2]^{\varepsilon_{n2}} \cdots [x_n, x_{n-1}]^{\varepsilon_{n,n-1}} \end{cases}$$

So the condition for $\phi(w)$ to be equal to 1 is that all exponents are equal to 0, hence $w = 1$.

□

Remark 4.7. As a consequence of the previous calculation we have that the homomorphism ϕ coincides on UVP_n with the abelianization map.

As a consequence of Proposition 4.6, representation ϕ is not faithful. However, according to previous characterization of WB_n as subgroup of $\mathrm{Aut}(F_n)$ it is natural to ask if we can give a characterization of automorphisms of $\mathrm{Aut}(N_n)$ that belong to $\phi(UVB_n)$.

Proposition 4.8. *Let β be an element of $\mathrm{Aut}(N_n)$, then $\beta \in \phi(UVB_n)$ if and only if β satisfies the condition $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$ with $1 \leq i \leq n$, where $\pi \in S_n$ and $a_i \in N_n$.*

Proof. Let us denote with $UVB(N_n)$ the subgroup of $\mathrm{Aut}(N_n)$ such that any element $\beta \in UVB(N_n)$ has the form $\beta(x_i) = g_i^{-1} x_{\pi(i)} g_i$, denoted by $x_{\pi(i)}^{g_i}$, with $1 \leq i \leq n$, where $\pi \in S_n$ and $g_i \in N_n$. We need to prove that $\phi \colon UVB_n \to UVB(N_n)$ is an epimorphism. Let β be an element of $UVB(N_n)$. Since S_n is both isomorphic to the subgroup of

UVB_n generated by the ρ_i generators, and to the subgroup of $UVB(N_n)$ generated by the permutation automorphisms, we can assume that for β the permutation is trivial, *i.e.*, $\beta(x_i) = x_i^{g_i}$. We define $\varepsilon_{i,j}$ to be $\phi(\lambda_{i,j})$ as in Proposition 4.6, and we prove that β is a product of such automorphisms. We recall that $x^{yz} = x^{zy}$ for any $x, y, z \in N_n$, therefore

$$\beta(x_i) = x_i^{x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}}$$

where $a_{i,i} = 0$.

In particular we can assume that

$$\beta(x_1) = x_1^{x_2^{a_{1,2}} \cdots x_n^{a_{1,n}}}.$$

We define a new automorphism β_1 multiplying β by $\varepsilon_{1,2}^{-a_{1,2}} \cdots \varepsilon_{1,n}^{-a_{1,n}}$. We have that $\beta_1(x_1) = x_1$, and $\beta_1(x_j) = \beta(x_j)$ for $j \neq 1$. Then again we define a new automorphism $\beta_2 = \beta_1\ \varepsilon_{1,2}^{-a_{2,1}} \varepsilon_{2,3}^{-a_{2,3}} \cdots \varepsilon_{2,n}^{-a_{2,n}}$ that fixes x_1 and x_2.

Carrying on in this way for n steps we get to an automorphism

$$\beta_n = \beta_{n-1}\ \varepsilon_{n,1}^{-a_{n,1}} \cdots \varepsilon_{n,n-1}^{-a_{n,n-1}} = \beta\ \prod_{j=1}^{n} \varepsilon_{n,j}^{-a_{1,j}} \prod_{j=1}^{n} \varepsilon_{n-1,j}^{-a_{2,j}} \cdots \prod_{j=1}^{n} \varepsilon_{1,j}^{-a_{n,j}}$$

setting $\varepsilon_{i,i} = 1$. The automorphism β_n is the identity automorphism: then β is a product of $\varepsilon_{i,j}$ automorphisms, hence it has a pre-image in UVB_n. □

4.2. **The fused link group.** Let L be a fused link. Then there exists an unrestricted virtual braid β such that its closure $\hat{\beta}$ is equivalent to L.

Definition 4.9. The fused link group $G(L)$ is the group given by the presentation

$$\left\langle x_1, \dots, x_n \;\middle|\; \begin{array}{ll} \phi(\beta)(x_i) = x_i & \text{for } i \in \{1, \dots, n\}, \\ [x_i, [x_k, x_l]] = 1 & \text{for } i, k, l \text{ not necessarily distinct} \end{array} \right\rangle$$

where $\phi \colon UVB_n \to \mathrm{Aut}(N_n)$ is the map from Proposition 4.5.

Proposition 4.10. *The fused link group is invariant under fused isotopy.*

Proof. According to [23] two unrestricted virtual braids have fused isotopic closures if and only if they are related by *braid moves* and *extended Markov moves*. We should check that under these moves the fused link group $G(L)$ of a fused link L does not change. This is the case. However a quicker strategy to verify the invariance of this group is to remark that it is a projection of the *welded link group* defined in [6, Section 5]. This last one being an invariant for welded links, we only have to do the verification for the second forbidden braid move, coming from relation (F2). This invariance is guaranteed by the fact that ϕ preserves relation (F2) as seen in Proposition 4.5. □

Remark 4.11. Let us recall that, according Theorem 3.6, a fused link L admits as a representative the closure of an element of UVP_n, say β_L, and following the proof of Proposition 4.8, we can deduce that

$$\phi(\beta(x_i)) = x_i^{x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}}$$

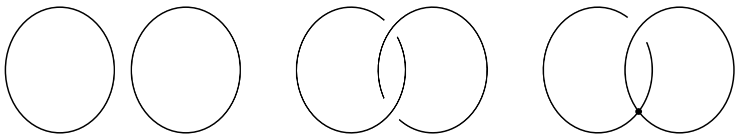

FIGURE 11. The group distinguishes the unlink U_2 from the Hopf link H, but does not distinguish the Hopf link with two classical crossings H from the one with a classical and a virtual crossing H_1. In fact: $G(U_2) = N_2$, while $G(H) = G(H_1) = \mathbb{Z}^2$. We remark however that H and H_1 are distinguished by the virtual linking number.

where $a_{i,i} = 0$ and $a_{i,j} = vlk_{i,j}$ for $i \neq j$; Since virtual linking numbers are fused invariants, we get another easy proof of Proposition 4.10. However, it means also that the knot group is determined by virtual linking numbers; actually, as shown in Figure 11, is weaker. The relation between virtual linking numbers and the knot group can be nicely described in the case $n = 2$ as follows. Let us consider $\lambda_{1,2}^\alpha \lambda_{2,1}^\beta$ and $\lambda_{1,2}^\gamma$, where γ is the greatest common divisor of α and β and therefore of $vlk_{1,2}$ and $vlk_{2,1}$. The automorphisms associated to them are

$$\phi(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) : \begin{cases} x_1 \longmapsto x_1\big[x_1, x_2[x_1,x_2]^{-\beta}\big]^\alpha = x_1\big[x_1, [x_1,x_2]^{-\beta}\big]^\alpha [x_1,x_2]^\alpha = x_1[x_1,x_2]^\alpha \\ x_2 \longmapsto x_2[x_1,x_2]^{-\beta}; \end{cases}$$

$$\phi(\lambda_{1,2}^\gamma) : \begin{cases} x_1 \longmapsto x_1[x_1,x_2]^\gamma \\ x_2 \longmapsto x_2. \end{cases}$$

Then

$$G(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) = G(\lambda_{1,2}^\gamma) = \langle x_1, x_2 \mid [x_1,x_2]^\gamma = 1,\ [x_i,[x_k,x_l]] = 1 \text{ for } i,k,l \in \{1,2\} \rangle =$$

$$= \langle x_1, x_2, t \mid [x_1,x_2] = t, t^\gamma = 1, t \text{ central} \rangle$$

This latter group presentation allows to distinguish these groups for different $\gamma \in \mathbb{N}$ (since γ is the order of the central element t of these Heisenberg-like groups, setting that $\gamma = 0$ means that t has infinite order); in particular we can set $G(\lambda_{1,2}^\gamma) := G_\gamma$. For instance the two links considered in [13], $L = \widehat{\sigma_1\rho_1\sigma_1^{-1}\rho_1}$ and U_2, have corresponding groups $G_1 = \mathbb{Z}^2$ and $G_0 = N_2$ and therefore are distinguished by G_γ, while, as we saw above, have the same classical linking number.

5. Other quotients

Several other quotients of virtual braid groups have been studied in the literature: we end this paper with a short survey on them, giving the structure of the corresponding pure subgroups and some results on their linearity.

5.1. **Flat virtual braids.** The study of flat virtual knots and links was initiated by Kauffman [20] and their braided counterpart was introduced in [21]. The category of flat virtual knots is identical to the structure of what are called virtual strings by V. Turaev in [29] (remark that every virtual string is the closure of a flat virtual braid).

The flat virtual braid grous FVB_n was introduced in [21] as a quotient of VB_n adding relations

$$\sigma_i^2 = 1, \quad \text{for } i = 1, \ldots, n-1. \tag{8}$$

It is evident that FVB_n is a quotient of the free product $S_n * S_n$.

Let us consider the natural projection map $f\colon VB_n \to FVB_n$, and set $f(\rho_i) := \rho_i$ and $f(\sigma_i) := s_i$ for $i = 1, \ldots, n-1$.

In addition to relations coming from the two copies of S_n, in FVB_n we have mixed relations

$$s_i \rho_j = \rho_j s_i, \quad \text{for } |i-j| \geq 2, \tag{9}$$

$$\rho_i \rho_{i+1} s_i = s_{i+1} \rho_i \rho_{i+1}, \quad \text{for } i = 1, \ldots, n-2. \tag{10}$$

We call *flat virtual pure braid group* FVP_n the kernel of the map $FVB_n \to S_n$ defined by $s_i, \rho_i \mapsto (i, i+1)$ for $i = 1, \ldots, n-1$. With respect to the map $f\colon VB_n \to FVB_n$, we have that $f(VP_n) = FVP_n$.

Proposition 5.1. *Let VP_n^+ be the (abstract) presented group*

$$\left\langle \{\lambda_{i,j} \mid 1 \leq i < j \leq n\} \;\middle|\; \begin{array}{c} \lambda_{k,l} = \lambda_{k,l}\lambda_{i,j}, \\ \lambda_{k,i}(\lambda_{k,j}\lambda_{i,j}) = (\lambda_{i,j}\lambda_{k,j})\lambda_{k,i} \end{array} \right\rangle.$$

Then VP_n^+ coincides with the subgroup of VP_n generated by the set $\{\lambda_{i,j} \mid 1 \leq i < j \leq n\}$ and is isomorphic to FVP_n.

Proof. First let us recall that VP_n is generated by elements $\lambda_{i,j}$ defined in Eq. (1) and (2), and has the following complete set of relations:

$$\lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j}, \tag{RS1}$$

$$\lambda_{k,i}(\lambda_{k,j}\lambda_{i,j}) = (\lambda_{i,j}\lambda_{k,j})\lambda_{k,i}. \tag{RS2}$$

Now define the map $\iota\colon VP_n^+ \to VP_n$ sending $\lambda_{i,j}$ to $\lambda_{i,j}$ and the map $\theta\colon VP_n \to VP_n^+$ sending $\lambda_{i,j}$ to $\lambda_{i,j}$ if $i < j$ or to $\lambda_{j,i}^{-1}$ whenever $i > j$. Both ι and θ are well defined homorphisms and $\theta \circ \iota = Id_{VP_n^+}$ so ι is injective.

Setting $f(\lambda_{i,j}) = \mu_{i,j}$ and proceeding with similar arguments as in Theorem 2.7 one can easily prove that FVP_n admits the presentation:

$$\left\langle \{\mu_{i,j} \mid 1 \leq i \neq j \leq n\} \;\middle|\; \begin{array}{c} \mu_{i,j}\mu_{k,l} = \mu_{k,l}\mu_{i,j}, \\ \mu_{k,i}(\mu_{k,j}\mu_{i,j}) = (\mu_{i,j}\mu_{k,j})\mu_{k,i}, \\ \mu_{i,j}\mu_{j,i} = 1 \quad \text{for } 1 \leq i \leq n-1 \end{array} \right\rangle.$$

We can proceed as before and to consider the abstract group FVP_n^+ given by following presentation:

$$\left\langle \{\mu_{i,j} \mid 1 \leq \underline{i < j} \leq n\} \;\middle|\; \begin{array}{c} \mu_{i,j}\mu_{k,l} = \mu_{k,l}\mu_{i,j} \\ \mu_{k,i}(\mu_{k,j}\mu_{i,j}) = (\mu_{i,j}\mu_{k,j})\mu_{k,i} \end{array} \right\rangle.$$

We can therefore consider map $\iota'\colon FVP_n^+ \to FVP_n$ sending $\mu_{i,j}$ to $\mu_{i,j}$ and the map $\theta'\colon FVP_n \to FVP_n^+$ sending $\mu_{i,j}$ to $\mu_{i,j}$ if $i < j$ or to $\mu_{j,i}^{-1}$ whenever $i > j$. Both ι' and θ' are well defined homeomorphisms and $\theta' \circ \iota' = Id_{FVP_n^+}$ and $\iota' \circ \theta' = Id_{FVP_n}$. Then FVP_n^+ is a group presentation for FVP_n and the isomorphism of the statement is obviously obtained sending $\mu_{i,j}$ to $\lambda_{i,j}$. □

Remark 5.2. For $n = 3$, the group

$$FVP_3 = \langle \lambda_{1,2}, \lambda_{1,3}, \lambda_{2,3} \mid \lambda_{1,2}^{-1}(\lambda_{2,3}\lambda_{1,3})\lambda_{1,2} = \lambda_{1,3}\lambda_{2,3} \rangle$$

is the HNN-extension of the free group $\langle \lambda_{1,3}, \lambda_{2,3} \rangle$ of rank 2 with stable element $\lambda_{1,2}$ and with associated subgroups $A = \langle \lambda_{2,3}\lambda_{1,3} \rangle$ and $B = \langle \lambda_{1,3}\lambda_{2,3} \rangle$, which are isomorphic to the infinite cyclic group. Moreover, the group FVP_3 is isomorphic to the free product $\mathbb{Z}^2 * \mathbb{Z}$. The first claim follows from the previous theorem. The second one follows from the observation that setting $a = \lambda_{23}\lambda_{13}$, $b = \lambda_{23}$ we obtain the following new presentation:

$$FVP_3 = \langle \lambda_{12}, a, b \mid\mid \lambda_{12}^{-1} a \lambda_{12} = b^{-1}ab \rangle;$$

if we denote $c = b\lambda_{12}^{-1}$ and exclude λ_{12} from the set of generators we get

$$FVP_3 = \langle a, b, c \mid\mid [a, c] = 1 \rangle = \langle a, c \mid [a, c] = 1 \rangle * \langle b \rangle.$$

Let us recall that there is another remarkable surjection of the virtual braid group VB_n onto the symmetric group S_n, which sends σ_i into 1 and ρ_i into $(i, i+1)$: the kernel of this map is denoted by H_n in [5]. In the same way we can define the group FH_n as the kernel of the homomorphism $\mu\colon FVB_n \to S_n$, which is defined as follows:

$$\mu(s_i) = 1,\ \mu(\rho_i) = (i, i+1),\ i = 1, 2, \ldots, n-1.$$

Now let us define, for $i = 1, \ldots, n-1$:

$$\begin{aligned} y_{i,i+1} &= s_i, \\ y_{i+1,i} &= \rho_i\, s_i \rho_i\,. \end{aligned} \tag{11}$$

For $1 \leq i < j-1 \leq n-1$:

$$\begin{aligned} y_{i,j} &= \rho_{j-1} \cdots \rho_{i+1} s_i \rho_{i+1} \cdots \rho_{j-1}, \\ y_{j,i} &= \rho_{j-1} \cdots \rho_{i+1} \rho_i\, s_i \rho_i\, \rho_{i+1} \cdots \rho_{j-1}. \end{aligned} \tag{12}$$

It is not difficult to prove that these elements belong to FH_n and that:

Theorem 5.3. *The group FH_n admits a presentation with generators $y_{k,l}$, for $1 \leq k \neq l \leq n$, and defining relations:*

$$y_{k,l}^2 = 1, \tag{13}$$

$$y_{i,j}\, y_{k,l} = y_{k,l}\, y_{i,j} \Leftrightarrow (y_{i,j}\, y_{k,l})^2 = 1, \tag{14}$$

$$y_{i,k}\, y_{k,j}\, y_{i,k} = y_{k,j}\, y_{i,k}\, y_{k,j} \Leftrightarrow (y_{i,k}\, y_{k,j})^3 = 1, \tag{15}$$

where distinct letters stand for distinct indices.

Proof. We can use Reidemeister-Schreier method and check the above set of relations is complete or we can consider a commutative diagram similar to the one of proof of Theorem 2.7:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker f_{|H_n} & \longrightarrow & \ker f & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & H_n & \longrightarrow & VB_n & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle f_{|H_n}} & & \downarrow{\scriptstyle f} & & \| & & \\
1 & \longrightarrow & FH_n & \longrightarrow & FVB_n & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Recall also that, according to [5], the group H_n is generated by following elements:

$$
\begin{aligned}
h_{i,i+1} &= \sigma_i\,, \\
h_{i+1,i} &= \rho_i\,\sigma_i\,\rho_i\,,
\end{aligned} \tag{16}
$$

and for $1 \leq i < j-1 \leq n-1$:

$$
\begin{aligned}
h_{i,j} &= \rho_{j-1}\cdots\rho_{i+1}\sigma_i\,\rho_{i+1}\cdots\rho_{j-1}, \\
h_{j,i} &= \rho_{j-1}\cdots\rho_{i+1}\rho_i\,\sigma_i\,\rho_i\,\rho_{i+1}\cdots\rho_{j-1},
\end{aligned} \tag{17}
$$

with defining relations:

$$h_{i,j}\,h_{k,l} = h_{k,l}\,h_{i,j}, \tag{18}$$

$$h_{i,k}\,h_{k,j}\,h_{i,k} = h_{k,j}\,h_{i,k}\,h_{k,j}, \tag{19}$$

where distinct letters stand for distinct indices. Now, remarking first that $f(h_{i,j}) = y_{i,j}$, $\ker f = \ker f_{|H_n} = \ll \sigma_i^2 \mid i = 1, 2, \ldots, n-1 \gg$ and $\sigma_i^2 = h_{i,i+1}^2$, one can also verify that $\ker f_{|H_n}$ is generated by elements of type $gh_{k,l}^2g^{-1}$, for $1 \leq k \neq l \leq n$ and $g \in H_n$ (details are left to the reader, but arguments are the same as in Theorem 2.7). Therefore a complete set of relations for FH_n is the following:

$$y_{k,l}^2 = 1, \tag{20}$$

$$y_{i,j}\,y_{k,l} = y_{k,l}\,y_{i,j}, \tag{21}$$

$$y_{i,k}\,y_{k,j}\,y_{i,k} = y_{k,j}\,y_{i,k}\,y_{k,j}. \tag{22}$$

□

Corollary 5.4. *The group FVB_n is linear.*

Proof. From the decomposition $VB_n = H_n \rtimes S_n$ we have that $FVB_n = FH_n \rtimes S_n$, where FH_n is a finitely generated Coxeter group. The statement therefore follows from the fact that all finitely generated Coxeter groups are linear and that finite extensions of linear groups are also linear. □

5.2. **Flat welded braids.** In a similar way we can define the flat welded braid group FWB_n as the quotient of WB_n adding relations

$$\sigma_i^2 = 1, \quad \text{for } i = 1, \ldots, n-1. \tag{23}$$

Let us consider the natural projection map $g\colon VB_n \to FVB_n$, and set $g(\rho_i) = \rho_i$ and $g(\sigma_i) = s_i$ for $i = 1, \ldots, n-1$.

In FWB_n, in addition to relations (9) and (10), we also have relations coming from relations of type (F1), *i.e.*,

$$s_{i+1} s_i \rho_{i+1} = \rho_i s_{i+1} s_i, \quad \text{for } i = 1, \ldots, n-1. \tag{24}$$

In FWB_n relations (23) and (24) imply that also relations of type (F2) hold, since from $\rho_i s_{i+1} s_i = s_{i+1} s_i \rho_{i+1}$ one gets $s_i s_{i+1} \rho_i = \rho_{i+1} s_i s_{i+1}$.

Adapting Theorem 2.7 one can easily verify that FWP_n is isomorphic to $\mathbb{Z}^{n(n-1)/2}$. As a straightforward consequence of Theorem 2.4, we can describe the structure of FWB_n.

Proposition 5.5. *Let $\mathbb{Z}^{n(n-1)/2}$ be the free abelian group of rank $n(n-1)/2$. Let us denote by $x_{i,j}$ for $1 \leq i \neq j \leq n$ a possible set of generators. The group FWB_n is isomorphic to $\mathbb{Z}^{n(n-1)/2} \rtimes S_n$, where S_n acts by permutation on the indices of generators of $\mathbb{Z}^{n(n-1)/2}$ (setting $x_{j,i} := x_{i,j}^{-1}$ for $1 \leq i < j \leq n$).*

Proof. Let us recall how elements $\lambda_{i,j}$ in UVB_n were defined.

For $i = 1, \ldots, n-1$:

$$\begin{aligned} \lambda_{i,i+1} &= \rho_i \sigma_i^{-1}, \\ \lambda_{i+1,i} &= \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i. \end{aligned}$$

For $1 \leq i < j-1 \leq n-1$:

$$\begin{aligned} \lambda_{i,j} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}, \\ \lambda_{j,i} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}. \end{aligned}$$

Relations (23) are therefore equivalent to relations $\lambda_{i,j} \lambda_{j,i} = 1$. Adding these relations and following verbatim the proof of Theorem 2.7 we get the statement. □

5.3. **Virtual Gauss braids.** From the notion of flat virtual knot we can get the notion of Gauss virtual knot or simply Gauss knot. Turaev [30] introduced these knots under the name of "homotopy classes of Gauss words", while Manturov [26] used the name "free knots".

The "braided" analogue of Gauss knots, called *free virtual braid group on n strands*, was introduced in [27]. From now on we will be calling it *virtual Gauss braid group* and will denote it by GVB_n.

The group of *virtual Gauss braids* GVB_n is the quotient of FVB_n by relations

$$s_i \rho_i = \rho_i s_i, \quad \text{for} \quad i = 1, \ldots, n-1.$$

Note also that the virtual Gauss braid group is a natural quotient of the *twisted virtual braid group*, studied for instance in [24].

Once again we can consider the homomorphism from GVB_n to S_n that sends each generator s_i and ρ_i in ρ_i. The *virtual Gauss pure braid group* GVP_n is defined to be the kernel of this map. Since this map admits a natural section GVB_n is isomorphic to $GVP_n \rtimes S_n$.

Adapting the proof of Theorem 2.7, we get the following.

Proposition 5.6. *The group GVP_n admits a presentation with generators $\lambda_{k,l}$ for $1 \leq k < l \leq n$ and the defining relations of FVP_n plus relations*

$$\lambda_{i,j}^2 = 1, \quad \textit{for} \quad 1 \leq i < j \leq n.$$

Moreover as in the case of FVB_n also in the case of GVB_n we can consider the map $\mu : GVB_n \to S_n$, defined as follows:

$$\mu(s_i) = 1,\ \mu(\rho_i) = \rho_i, \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

Let GH_n be the kernel of the map $\mu : GVB_n \to S_n$, and $y_{k,l}$, the elements defined in subsection 5.1: we can prove the following result.

Theorem 5.7. *The group GH_n admits a presentation with generators $y_{k,l}$, $1 \leq k < l \leq n$, and defining relations:*

$$y_{k,l}^2 = 1, \tag{25}$$

$$(y_{i,j}\, y_{k,l})^2 = 1, \tag{26}$$

$$(y_{i,k}\, y_{k,j})^3 = (y_{i,j}\, y_{k,j})^3 = (y_{i,k}\, y_{i,j})^3 = 1, \tag{27}$$

where distinct letters stand for distinct indices.

Proof. We leave the proof to the reader, since one can follow the same approach as in Theorems 2.7 and 5.3. The key point is that GVB_n is the quotient of FVB_n by the set of relations

$$s_i \rho_i = \rho_i s_i, \quad i = 1, 2, \ldots, n-1.$$

One can easily verify that it implies that $y_{j,i} = y_{i,j}$, for $1 \leq i < j \leq n$. Hence, GH_n is generated by elements $y_{k,l}$, for $1 \leq k < l \leq n$. If we rewrite the set of relations of FH_n in these generators and we proceed as in Proposition 5.1 we get the set of relations given in the statement. As before, one can also use the Reidemeister-Schreier method to check that this is a complete set of relations. □

As corollary, we have:

Corollary 5.8. *The group GVB_n is linear.*

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