Topological generalizations of braid groups: combinatoric properties and applications to knot theory

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Remerciements

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Introduction

Braids are ubiquitous objects than can be defined in several different ways, each leading deep insights and possible applications in many fields. Plenty of good surveys are available on the (recent) literature for instance [?], or [?], which focuses more particularly on computational problems, or [?] devoted to algebraical generalizations, namely Artin and Garside groups.

In this work we will give just few remainders of basic equivalent definition of braid groups and to introduce some of their topological (or "diagrammatical") generalizations and we will focus mainly the context of my research, the results obtained and possible perspectives.

In the first Chapter we will therefore recall some equivalent definitions of braid groups, in terms of mapping classes, configuration spaces, collections of paths, automorphisms of free groups: this will allow us to introduce some generalizations: topological (surface braid groups), as automorphisms of the free group (welded braid groups) or diagrammatical (virtual braid groups). Along the Chapters we will show that also these generalizations have several equivalent definitions (Chapter 1 and 4 in particular). Other algebraical generalizations, namely Artin-Tits groups, will be recalled in Chapter 2 and 4.

The Chapters 2–6 describe the main results from selected works [B1] – [B16]. In Chapter 2 we give a survey on group presentations of surface braid groups and we present main results of [?, ?, ?]. We present shortly group presentations provided in [?]. We show also some positive presentations for surface braid groups given in [?] and presentations by graphs provided in [?] describing completely only the case of the sphere.

The Chapter 3 deals with the notion of singular generalized braids and associated finite type invariant theory. More precisely, the first aim of Chapter 3 is to introduce the notion of singular surface braid monoids and present results from [?], in particular the geometrical characterization of *quasi*-centralizers of generators of these monoids; we will show also that one can deduce from this result that the word problem is solvable for singular surface braid monoids. The second aim is to give a survey on finite type invariant theory for (surface) braids and to present main results from [?], stating that there is not an equivalent of Kontsevich integral for braid groups on surfaces of genus $g \geq 1$ and providing a possible new notion of finite type invariants. Finally we will present the notion of finite type invariants for virtual braids, proposed in [?] as a generalization of finite type invariants for virtual knots suggested in [?] and afterwards considered and widely studied in [?].

The Chapter 4 is devoted to present results from [?, ?, ?, ?]. In the first part of this Chapter 4 we give a survey on automorphisms of braid group of surfaces (with a particular attention to hopfian and cohopfian properties) and to present main results from [?], i.e. the determination of (outer) automorphism groups for braid groups of closed surfaces. In the second part we will extend representations of braids as automorphisms of free groups presented in Chapter 1 to several generalisations of braids studied in [?, ?, ?]: Artin-Tits groups, surface braid group and virtual braid groups. We end showing an application to virtual (and welded) knot theory constructing virtual (and welded) knot invariants [?] and providing some possible perspectives.

In Chapter 5 we determine completely lower central series of surface braid groups and of their pure subgroups ([**?**, **?**]); in addition to their relevance in finite type invariant theory, we will show also some possible applications in the study of linearity of these groups ([**?**]). We provide also a short survey on exact sequences for pure and mixed surface braid groups and we present the notion of surface framed braid group, introduced and studied in [**?**].

Finally in Chapter 6 we give a short introduction on relations between (surface) braids and links in 3-manifolds and we present the notion of Hilden surface braid groups, introduced in [**?**] and related to a possible generalization of the notion of plat closure : we provide a infinite set of generators for these groups and we present some possible applications and perspectives in the study of links in 3-manifolds.

All colored pictures are screenshots of *Braids: a movie*, a computer graphics animation of my student Ester Dalvit. The movie is available on *http://http://matematita.science.unitn.it/braids/* : more information about this project can be founded in the PhD thesis of Ester Dalvit [**?**].

Chapter 1

Braids and their topological generalizations

1.1 Braid groups

The braid group B_n, originally introduced by Artin in 1925, can be defined in several ways: a short history of the different definitions of braid groups can be founded for instance in [**?**].

Braids as mapping classes. The first definition that we will recall is in terms of mapping classes of the punctured disk. Specifically, let $\mathbb{D}^2 = \{z \in \mathbb{R}^2 \,|\, |z| \leq 1\}$ be the unit disc with counterclockwise orientation. Fix a set of $n \geq 1$ distinct punctures $\mathcal{P} = \{p_1, ..., p_n\} \subset \mathrm{int}(\mathbb{D}^2)$. We shall assume that $p_1 < ... < p_n \in (-1, +1) = \mathbb{R} \times \{0\} \cap \mathrm{int}(\mathbb{D}^2)$. Set $\mathbb{D}_n = \mathbb{D}^2 \backslash \mathcal{P}$. The group $\pi_0(\mathrm{Homeo}(\mathbb{D}_n))$ of isotopy classes of those (orientation preserving) homeomorphisms of D_n that fix pointwise the unit circle $\partial\mathbb{D}_n$ is denoted B_n and called the *n-th braid group.*

For $i = 1, ..., n-1$, the linear interval $[p_i, p_{i+1}] \subset (-1, +1) \subset \mathbb{R}$ is an embedded arc in $\mathbb{D}^2$ with endpoints in the punctures p_i, p_{i+1}. Consider a homeomorphism that permutes counterclockwisely the punctures p_i and p_{i+1} by using a rotation of order π and it is the identity outside a neighborhood of the disk of diameter p_ip_{i+1}. The mapping class corresponding to such a homeomorphism is uniquely determined and also called a Dehn half-twist (or braid twist) $\mathbb{D}_n \longrightarrow \mathbb{D}_n$, that we denote by σ_i. It is a classical fact that B_n is generated by $\sigma_1, ..., \sigma_{n-1}$ with defining relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ for } |i-j| \geq 2$$

and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \text{ for } i = 1, ..., n-2$$

Let $\mathfrak{S}_n$ be the symmetric group on n elements. There is a natural map $\pi : B_n \longrightarrow \mathfrak{S}_n$ which associates to any element of B_n the corresponding permutation on the set $\mathcal{P}$. The image of σ_i in $\mathfrak{S}_n$ is the transposition $(i, i+1)$ and the kernel of π is usually called pure braid group on n strands and denoted by P_n. Otherwise stated, P_n is the subgroup of B_n consisting of isotopy classes of homeomorphisms fixing point-wise the set $\mathcal{P}$.

Braids as collection of paths. Another definition of B_n can be given in terms of collections of paths. A *geometric braid* based at $\mathcal{P}$ is an n-tuple $\beta = (\psi_1, \ldots, \psi_n)$ of paths (*strands*) $\psi_i : [0,1] \longrightarrow \mathbb{D}^2 \times [0,1]$ such that

- $\psi_i(0) = p_i \times 0,\ i = 1, \ldots, n$;
- $\psi_i(1) \in \mathcal{P} \times 1,\ i = 1, \ldots, n$;
- $\psi_1(t), \ldots, \psi_n(t)$ belong to $\mathbb{D}^2 \times \{t\}$ and are distinct points of all $t \in [0, 1]$.

The usual product of paths induces a group structure on the set of braids up to homotopies among braids. This group does not depend on the choice of $\mathcal{P}$ and it is isomorphic to B_n.

The equivalence between these two definitions of B_n is (roughly) established as follows. Any homeomorphism $h : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ which fixes $\partial\mathbb{D}^2$ point-wise is related to the identity map $\mathrm{id}_{\mathbb{D}^2}$ by an isotopy $\{h_s : \mathbb{D}^2 \longrightarrow \mathbb{D}^2\}_{s\in[0,1]}$ such that $h_0 = \mathrm{id}_{\mathbb{D}^2}$ and $h_1 = h$. If $h(\mathcal{P}) = \mathcal{P}$ then the set $\cup_{s\in[0,1]}(h_s(\mathcal{P}) \times s) \subset \mathbb{D}^2 \times [0, 1]$ is a braid. Its isotopy class depends only on the element of B_n represented by h. This establishes an isomorphism between B_n and the group of braids on n strands, which will denote also by B_n.

The generator $\sigma_i \in B_n$ corresponds therefore to the i-th elementary braid give in Figure **??**

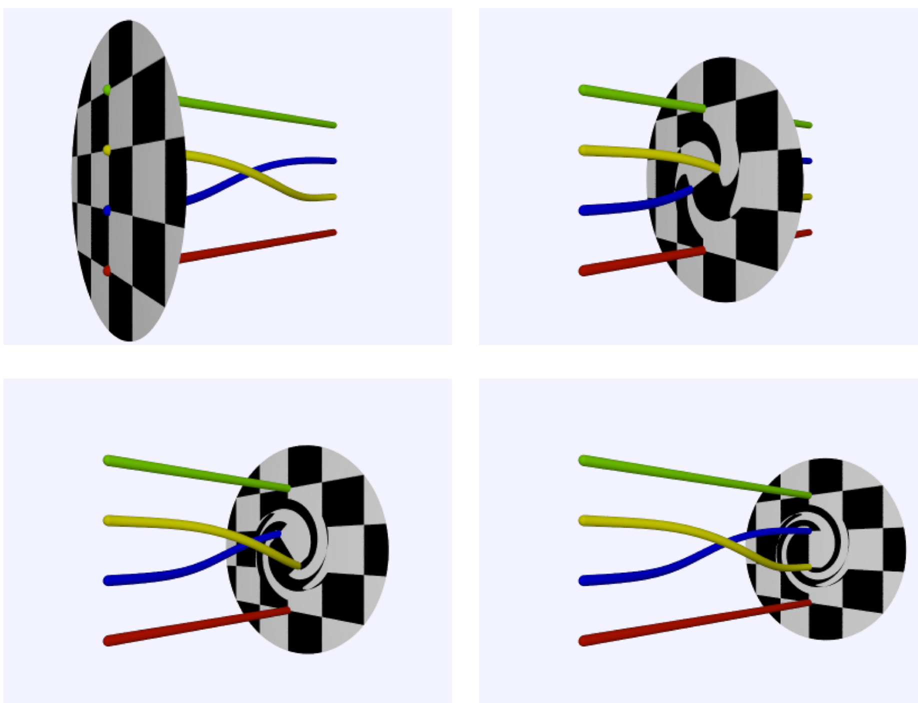


Figure 1.1: The braid twist is the homeomorphism that corresponds to a generator of the braid group.

Braids in terms of configuration spaces. A third topological definition of braid groups (which will be particularly useful in the following) can be given using configuration spaces Let $\mathbb{F}_n(\mathbb{C}) = \mathbb{C}^n \setminus \Delta$, where Δ is the fat diagonal, i.e. the set of n-tuples $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ for which $x_i = x_j$ for some $i \neq j$. The fundamental group $\pi_1(\mathbb{F}_n(\mathbb{C}))$ is isomorphic to the pure braid group on n strands, while $\pi_1(\mathbb{F}_n(\mathbb{C})/\mathfrak{S}_n)$, where $\mathfrak{S}_n$ acts on $F_n(\Sigma)$ by permutation of coordinates, is isomorphic to the braid group on n strands. We refer to Theorem 2.16 of [**?**] for an elegant proof of these isomorphisms : we will just remark that they can be deduced using the evaluation map

$Ev : \mathrm{Homeo}(\mathbb{D}_n) \longrightarrow \mathbb{F}_n(\mathbb{C})$ defined by $Ev(\varphi) = \{\varphi(p_1), \dots, \varphi(p_n)\}$ and the long homotopy exact sequence associated to this locally trivial fiber bundle.

The definition of B_n in terms of configurations spaces allows us to introduce the famous Fadell-Neuwirth fibration, i.e. the fibration $\mathbb{F}_{k+n}(\mathbb{C})/(\mathfrak{S}_k \times \mathfrak{S}_n) \longrightarrow \mathbb{F}_n(\mathbb{C})/\mathfrak{S}_n$ given by forgetting the first k coordinates is a locally-trivial fibration whose fibre over a point $\{x_1, \dots, x_n\}$ may be identified with the orbit space $\mathbb{F}_k(\mathbb{C} \backslash \{x_1, \dots, x_n\})/\mathfrak{S}_k$. From now on we denote $\pi_1(\mathbb{F}_{k+n}(\mathbb{C})/(\mathfrak{S}_k \times \mathfrak{S}_n))$ simply by $B_{k,n}$. Let us also denote $\pi_1(\mathbb{F}_k(\mathbb{C} \setminus \{x_1, \dots, x_n\})/\mathfrak{S}_k)$ by $B_k(\mathbb{D}_n)$; this group turns out to be isomorphic to the subgroup of B_{k+n} consisting of braids where the last n strands are trivial (vertical). The long exact sequence in homotopy of the above fibration yields a short exact sequence.

Lemma 1.1. *Let $k, n \in \mathbb{N}$. The Fadell-Neuwirth fibration $\mathbb{F}_{k+n}(\mathbb{C}) \longrightarrow \mathbb{F}_n(\mathbb{C})$ induces the short exact sequence:*

$$1 \longrightarrow B_k(\mathbb{D}_n) \longrightarrow B_{k,n} \longrightarrow B_n \longrightarrow 1\,. \tag{MB}$$

In a similar way, we may obtain the well-known short exact sequence of pure braid groups.

$$1 \longrightarrow P_k(\mathbb{D}_n) \longrightarrow P_{k+n} \longrightarrow P_n \longrightarrow 1. \tag{PB}$$

Here $P_k(\mathbb{D}_n)$ denotes the fundamental group of $\mathbb{F}_k(\mathbb{D}_n)$, which is isomorphic to the subgroup of P_{k+n} consisting of pure braids where the last n strands are vertical. Notice that the short exact sequences (MB) and (PB) split for all $k \geq 1$, where the section is given geometrically by adding k trivial strands 'at infinity' (see for instance Section 2.2 of [**?**]).

1.2 Representation of braids as automorphisms of free groups

We shall now give still another interpretation of braids. There is a natural representation of braids on n strands as automorphisms of the free group F_n of rank n. Although it is possible to explain this representation using (MB) sequence for $n = 1$ in Lemma **??**, it is probably more natural to define it by means of mapping classes, as was done (in a different language) by Magnus [**?**].

In fact, since B_n acts on $\mathbb{D}_n$, we obtain an action of B_n on the fundamental group of $\mathbb{D}_n$, which is a free group on n generators. This action is faithful and, once fixed $\{x_1, x_2, \dots, x_n\}$ as the set of generators of F_n, the braid group B_n may be represented as a subgroup of $\mathrm{Aut}(F_n)$ by associating to any (classical) generator σ_i, for $i = 1, 2, \dots, n-1$, of B_n the following automorphism of F_n:

$$\sigma_i : \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases}$$

Artin proved a stronger result (see [**?**, Theorem 5.1] for a nice and clear proof), by giving a characterization of braids as automorphisms of free groups. He proved that any automorphism β of $\mathrm{Aut}(F_n)$ corresponds to an element of B_n if and only if β satisfies the following conditions:

$$\begin{array}{rl} i) & \beta(x_i) = a_i^{-1}\, x_{s(i)}\, a_i, \quad 1 \leq i \leq n, \\ ii) & \beta(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n, \end{array}$$

where s is a permutation in the symmetric group $\mathfrak{S}_n$, and $a_i \in F_n$. As we will recall later, this representation has several straightforward consequences, such as the residually nilpotence of the pure braid group and some interesting applications: in particular given a braid β one can get this way a presentation for the group of the link isotopic to the Alexander closure of β (see Chapter 4)

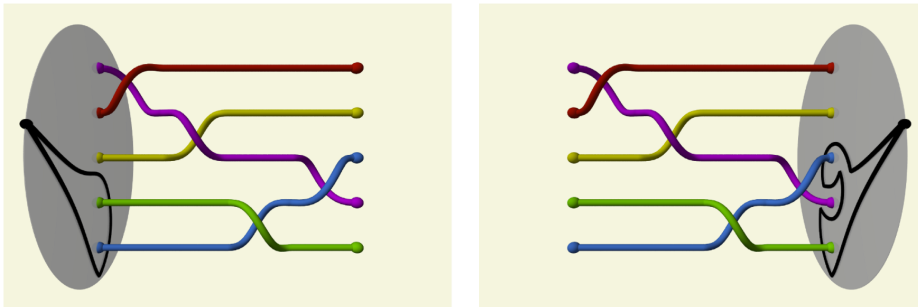


Figure 1.2: Braids on n strands act on $\pi_1(\mathbb{D}_n)$.

and one can define the (non reduced) Burau representation composing Artin representation with Magnus representation (see for instance [?]).

Generalizing Artin representation, in [?] Wada found several representations of B_n in $\mathrm{Aut}(F_n)$ which, with the usual braid closure (see for instance Chapter 6), provide group invariants of links. These representation are of the following special form: any generator σ_i of B_n acts trivially on generators of F_n except on the pair (x_i, x_{i+1}):

$$\begin{aligned} x_i \cdot \sigma_i &= u(x_i, x_{i+1})\,, \\ x_{i+1} \cdot \sigma_i &= v(x_i, x_{i+1})\,, \\ x_j \cdot \sigma_i &= x_j \qquad\quad j \neq i, i+1\,, \end{aligned}$$

where u and v are now words in the generators a, b, with $\langle a, b\rangle \simeq F_2$. Wada found seven families of such kind of representations explaining the geometrical meaning of related link invariants: the completeness of this set of classes of representations was recently proven by Ito [?].

In another direction Crisp and Paris considered Artin-type representations $\rho_{CP} : B_n \longrightarrow \mathrm{Aut}(H^{*n})$, where H^{*n} denotes the free product of n copies of a group H, exploring the associated link invariants and giving a necessary and sufficient criterion for $H^{*n} \rtimes_{\rho_{CP}} B_n$ to be a *Garside* group. In [?] is also given a classification up to equivalence of Wada representations.

Another interesting representation of B_n is the *braid monodromy representation* (also called Perron-Vannier representation) of B_n into $\mathrm{Aut}(F_{n-1})$: consider the surface $\Sigma_{g,b}$ of genus $g \geq 1$ and $1 \leq b \leq 2$ boundary components with the condition $2g + b = n$. This surface is the branched 2-fold cover of $\mathbb{C}$ branched over the set of n distinct points. The induced action on the fundamental group of $\Sigma_{g,b}$ defines therefore a representation $\rho_D : B_n \longrightarrow \mathrm{Aut}(F_{n-1})$ (for $n \geq 4$), given algebraically by:

$$\rho_D(\sigma_1) : \begin{cases} x_1 \longrightarrow x_1, \\ x_j \longrightarrow x_1^{-1} x_j, j \neq 1, \end{cases}$$

and for $2 \leq i \leq n-1$,

$$\rho_D(\sigma_i) : \begin{cases} x_{i-1} \longmapsto x_i, \\ x_i \longmapsto x_i x_{i-1}^{-1} x_i, \\ x_j \longmapsto x_j, \qquad\qquad j \neq i-1, i. \end{cases}$$

Starting from Perron-Vannier representation it is easy to construct another equivalent representation $\rho_{D'} : B_n \longrightarrow \mathrm{Aut} F_{n-1}$ given algebraically by:

$$\rho_{D'}(\sigma_i) : \begin{cases} g_{i-1} \longmapsto g_{i-1}g_i, & \\ g_{i+1} \longmapsto g_i^{-1}g_{i+1} & \\ g_j \longmapsto g_j, & j \neq i-1, i+1, \end{cases}$$

where $F_{n-1} = \langle g_1, \ldots, g_{n-1} \rangle$.

This representation $\rho_{D'}$ was proved to be faithful in [?] and [?]: the representation $\rho_{D'}$ appeared independently in [?] in the different framework of episturmian morphisms ([?]).

1.3 Surface braids

Surface braid groups are a natural generalization of the classical braid groups and of fundamental groups of surfaces.They were first defined by Zariski during the 1930's (braid groups on the sphere had been considered much earlier by Hurwitz, [?]), were re-discovered by Fox during the 1960's, and were used subsequently in the study of mapping class groups and of configuration spaces [?]. It is interesting to remark that this groups, introduced as an "algebraical" tool, turned out to be really difficult to understand and now one uses mapping class groups tools (like complex curves) to study the properties of automorphisms of surface braid groups (see for instance [?, ?] and Chapter 4).

In the last years the interest for these groups grew notably, in particular for their relations with knot theory ([?, ?, ?] and Chapter 6) and the structure of Lie Algebras related to pure braids ([?, ?, ?]). The finite type invariant theory for braids (see Chapter 3) have been extended to surface braids [?, ?, ?] and new features have been discovered [?, ?]. On the other hand the combinatoric of these groups is different and sometimes much more complicated that usual braid groups: for instance braid groups of surfaces are cohopfian ([?] and Chapter 4), the structure of pure braid subgroups are different than in the classical case (see in particular [?] and Chapter 5) and the lower and derived central series are particularly rich and they depend on the number of strands and the genus of the surface (see in particular [?, ?, ?, ?] and Chapter 5).

Several problems remains however unsolved, such the linearity (except few cases, essentially the sphere and the projective plane, [?, ?, ?]) and conjugacy problem.

As usual braid group, surface braid groups admit several equivalent definitions: we will recall three possible definitions, while in next Chapters we will present main results that we obtained in the study of these group.

Surface braid groups via configuration space. Let Σ be a connected, orientable surface. Let $F_n(\Sigma) = \Sigma^n \setminus \Delta$, where Δ is the set of n-tuples $x = (x_1, \ldots, x_n)$ for which $x_i = x_j$ for some $i \neq j$. The fundamental group $\pi_1(F_n(\Sigma))$ is called the *pure braid group* on n strands of the surface Σ; it shall be denoted by $P_n(\Sigma)$. Consider the quotient space $F_n(\Sigma)/\mathfrak{S}_n$ where the symmetric group $\mathfrak{S}_n$ acts by permutation of coordinates. The fundamental group $\pi_1(F_n(\Sigma)/\mathfrak{S}_n)$ is called the *braid group* on n strands of the surface Σ; it shall be denoted by $B_n(\Sigma)$.

Surface braid groups as equivalence classes of geometric braids. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points (*punctures*) in the interior of Σ. A *geometric braid* on Σ based at $\mathcal{P}$ is a collection $\beta = (\psi_1, \ldots, \psi_n)$ of n disjoint paths (called *strands*) on $\Sigma \times [0,1]$ which run monotonically with $t \in [0,1]$ and such that $\psi_i(0) = (p_i, 0)$ and $\psi_i(1) \in \mathcal{P} \times \{1\}$. Two braids are considered to be equivalent if they are isotopic relatively to the base points. The usual product of paths defines a group structure on the equivalence classes of braids. This group, which is isomorphic to $B_n(\Sigma)$, does not depend on the choice of $\mathcal{P}$. A braid is said to be *pure* if $\psi_i(1) = (p_i, 1)$ for all $i = 1, \ldots, n$. The set of pure braids form a group isomorphic to $P_n(\Sigma)$.

Mapping class groups, bounding pair braids and pure braids Let $\Sigma_{g,b}$ an orientable surface of genus g and with b boundary components. Let $\mathrm{Diff}^+(\Sigma_{g,b})$ denote the group of orientation preserving diffeomorphisms of $\Sigma_{g,b}$ which are the identity on the boundary. Recall that the *mapping class group* of $\Sigma_{g,b}$, denoted $\Gamma_{g,b}$, is defined to be $\pi_0(\mathrm{Diff}^+(\Sigma_{g,b}))$, where $\mathrm{Diff}^+(\Sigma_{g,b})$ is equipped with the compact open topology. If the surface has empty boundary then we shall just write Γ_g.

Let $\mathcal{P} = \{x_1, \ldots, x_n\}$ be a set of n distinct points in the interior of the surface $\Sigma_{g,b}$.

Let us also introduce two different subgroups of $\mathrm{Diff}^+(\Sigma_{g,b})$: $\mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{P}) = \{h \in \mathrm{Diff}^+(\Sigma_{g,b}) \mid \exists \sigma \in \mathfrak{S}_n,\ h(p_i) = p_{\sigma(i)}\}$, and $\mathrm{Diff}^+(\Sigma_{g,b}, \underline{p}) = \{h \in \mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{P}) \mid h(p_i) = p_i\}$

We call *punctured mapping class group* of $\Sigma_{g,b}$ relative to $\mathcal{P}$ the group $\pi_0\mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{P})$. This group, denoted by $\Gamma^n_{g,b}$, does not depend on the choice of $\mathcal{P}$, but just on its cardinal. We define the *pure punctured mapping class group*, denoted by $\mathrm{P}\Gamma^n_{g,b}$, to be the $\pi_0(\mathrm{Diff}^+(\Sigma_{g,b}, \underline{p}))$, i.e. the subgroup of (isotopy classes of) preserving orientation homeomorphisms which fix the set $\mathcal{P}$ pointwise.

Braid groups of a surface are related to mapping class groups as follows:

Theorem 1.1. ([?]) *Let $n \geq 1$. Let $\psi_n : \Gamma^n_{g,b} \longrightarrow \Gamma_{g,b}$ and $\varphi_n : \mathrm{P}\Gamma^n_{g,b} \longrightarrow \Gamma_{g,b}$ be the homomorphisms induced by the forgetting map $\mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{P}) \longrightarrow \mathrm{Diff}^+(\Sigma_{g,b})$ and $\mathrm{Diff}^+(\Sigma_{g,b}, \underline{p}) \longrightarrow \mathrm{Diff}^+(\Sigma_{g,b})$.*
1) If $(g, b) \notin \{(0, 0), (1, 0)\}$, $\ker \psi_n$ and $\ker \varphi_n$ are respectively isomorphic to $B_n(\Sigma_{g,b})$ and $P_n(\Sigma_{g,b})$.
2) When $b = 0$ and $g \in \{0, 1\}$, $\ker \psi_n$ and $\ker \varphi_n$ are respectively isomorphic to $B_n(\Sigma_g)/Z(B_n(\Sigma_g))$ and $P_n(\Sigma_g)/Z(P_n(\Sigma_g))$ (where $Z(G)$ is the center of the group G).

We note T_C a Dehn twist along a simple closed curve C. Let C and D be two simple closed curves bounding an annulus containing the single puncture x_j. We shall say that the multitwist $T_C T_D^{-1}$ is a *j-bounding pair braid*.

In particular, if Σ is an orientable surface (possibly with boundary) of positive genus and different from the torus, the surface pure braid group $P_n(\Sigma)$ may be identified with the subgroup of $\mathrm{P}\Gamma^n_{g,b}$ generated by bounding pair braids (see for instance [**?**], where bounding pair braids are called spin-maps).

1.4 Virtual and welded objects

The usual way to describe a braid (as collection of paths) is to give its "blackboard" representation, i.e. a smooth immersion on $\mathbb{R}^2$ with only transversal double points that we call crossings (with the usual information on overpasses and underpasses). During last twenty years several "diagrammatical" generalizations of braid groups were defined and studied, adding to classical crossing new kinds of crossings: we can recall on one hand singular braids [**?**, **?**], related to finite type invariant theory (Chapter 3), and on the other hand virtual braids and welded braids that we present briefly here and in Chapters 4 and 5.

1.4.1 Virtual braids

Virtual braid groups can be defined in different ways, combinatorial [**?**, **?**] or topological [**?**, **?**]: in the following we introduce virtual braid groups just as quotients of the free product of a braid group and the corresponding symmetric group.

The virtual braid group VB_n can be indeed defined as the group generated by the elements σ_i, ρ_i, $i = 1, 2, \ldots, n-1$ with the defining relations:

$$\sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2,$$

$$\sigma_i\,\sigma_j = \sigma_j\,\sigma_i, \quad |i-j| \geq 2.$$

$$\rho_i\,\rho_{i+1}\,\rho_i = \rho_{i+1}\,\rho_i\,\rho_{i+1}, \quad i = 1, 2, \ldots, n-2,$$

$$\rho_i\,\rho_j = \rho_j\,\rho_i, \quad |i-j| \geq 2.$$

$$\rho_i^2 = 1, \quad i = 1, 2, \ldots, n-1;$$

$$\sigma_i\,\rho_j = \rho_j\,\sigma_i, \quad |i-j| \geq 2,$$

$$\rho_i\,\rho_{i+1}\,\sigma_i = \sigma_{i+1}\,\rho_i\,\rho_{i+1}, \quad i = 1, 2, \ldots, n-2.$$

The diagrammatic interpretation of virtual braids is evident: the generator σ_i is illustrated by the diagram on the left in Figure **??** and ρ_i is illustrated by the diagram with a "virtual" crossing on the right in Figure **??**: previous relations can be therefore seen as equivalences of diagrams (see for instance Figure **??**).

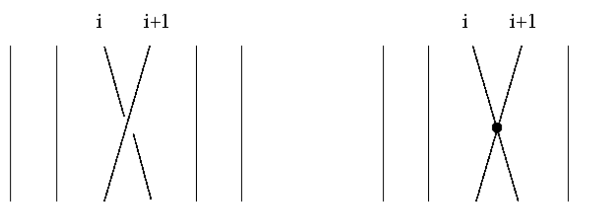


Figure 1.3: Interpretation of generators of VB_n

1.4.2 Welded braids

As we recalled in section **??** braids on n strands can be characterized as automorphisms of the free groupF_n satisfying the following conditions:

$$\begin{array}{ll} i) & \beta(x_i) = a_i^{-1}\,x_{\pi(i)}\,a_i, \quad 1 \leq i \leq n, \\ ii) & \beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n, \end{array}$$

where $\pi \in \mathfrak{S}_n$ and $a_i \in F_n$.

The group of conjugating automorphisms C_n consists of automorphisms satisfying the first condition. In [?] (see also [?] for a proof in terms of configuration spaces or [?] for a combinatorial proof based on results from [?]) it was proved that C_n has the following group presentation: it is generated by σ_i, α_i, $i = 1, 2, \ldots, n-1$, where elements of type σ_i generate the braid group B_n, elements of type α_i generate the symmetric group $\mathfrak{S}_n$, and the following mixed relations hold

$$\alpha_i\,\sigma_j = \sigma_j\,\alpha_i, \quad |i-j| \geq 2,$$

$$\alpha_{i+1}\,\alpha_i\,\sigma_{i+1} = \sigma_i\,\alpha_{i+1}\,\alpha_i, \quad i = 1, 2, \ldots, n-2,$$

$$\alpha_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\alpha_{i+1}, \quad i = 1, 2, \ldots, n-2.$$

The generators $\sigma_1, \ldots, \sigma_{n-1}$ of C_n correspond to previous automorphisms of F_n while any generator α_i, for $i = 1, 2, \ldots, n-1$ is associated to the following automorphism of F_n:

$$\alpha_i : \begin{cases} x_i \longmapsto x_{i+1} \\ x_{i+1} \longmapsto x_i, \\ x_l \longmapsto x_l, \qquad l \neq i, i+1. \end{cases}$$

The group C_n admits also other equivalent definition in terms of mapping classes, configuration spaces and therefore C_n appears often in the literature with different names and notations: group of flying rings [?, ?], McCool groups [?], motions groups [?]... According to [?] we will replace the notation C_n with WB_n and we will call this group *welded braid group on n strands.*

Comparing the defining relations of VB_n and WB_n, we see that the group presentation of WB_n can be obtained from the group presentation of VB_n replacing ρ_i by α_i and adding relations of type $\alpha_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \alpha_{i+1}, \quad i = 1, 2, \ldots, n-2$. As noticed by Kamada [?], the above representation of WB_n as conjugating automorphisms and the fact WB_n is a quotient of VB_n imply that the σ_i's generate the braid group B_n in VB_n.

1.4.3 Virtual and welded links

In a similar way we can generalize knot (link) diagrams to obtain singular,virtual or welded links. It is worth to mention that for all of these generalizations it exists an Alexander-like theorem, stating that any singular (respectively virtual or welded) link can be represented as the closure of a singular (respectively virtual or welded) braid. Moreover, there are generalizations of classical Markov's theorem for braids giving a characterization for two singular (respectively virtual or welded) braids whose closures represent the same singular (respectively virtual or welded) link (see [?, ?]).

In particular, virtual knot theory has been introduced by Kauffman [?] as a generalization of classical knot theory. Virtual knots (and links) are several geometric interpretations (see for instance [?, ?]) or combinatorial, in terms of Gauss codes ([?, ?]). Here we will limit to represent them as (equivalence classes of) generic immersions of circles in the plane (virtual link diagrams) where double points can be classical (with the usual information on overpasses and underpasses) or virtual.

Virtual link diagram are equivalent under ambient isotopy and some types of local moves (generalized Reidemeister moves): classical Reidemeister moves (Figure **??**), virtual Reidemeister moves and mixed Reidemeister moves (Figures **??** and **??**).

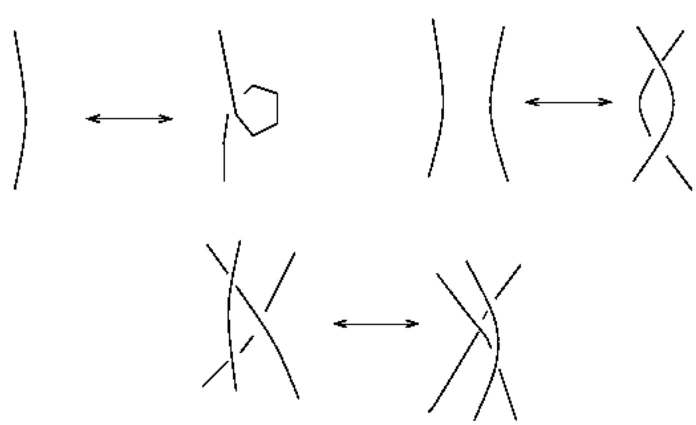


Figure 1.4: Classical Reidemeister moves

A Theorem of Goussarov, Polyak and Viro [?, Theorem 1B] states that if two classical knot diagrams are equivalent under generalized Reidemeister moves, then they are equivalent under the classical Reidemeister moves. In this sense virtual link theory is a nontrivial extension of classical theory. This Theorem is a straightforward consequence of the fact that the knot group (more precisely the *group system* of a knot, see for instance [?, ?]) provides a complete knot invariant

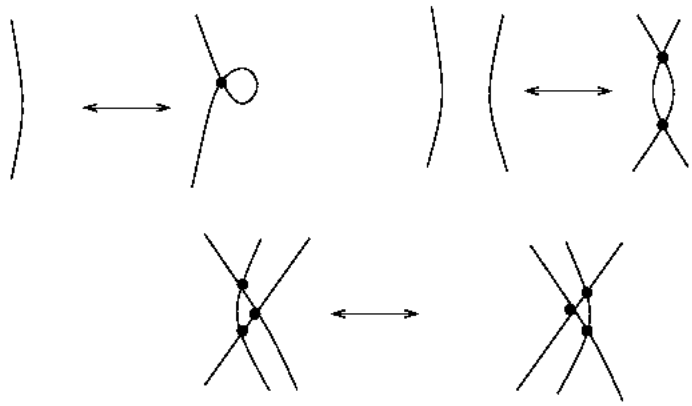


Figure 1.5: Virtual Reidemeister moves

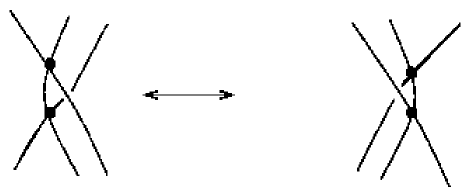


Figure 1.6: Mixed Reidemeister moves

which can be naturally extended in the realm of virtual links.

To the generalized Reidemeister moves on virtual diagrams one could add two other kinds of local moves, called *forbidden moves* of type $F1$ and $F2$ (Figure **??**).

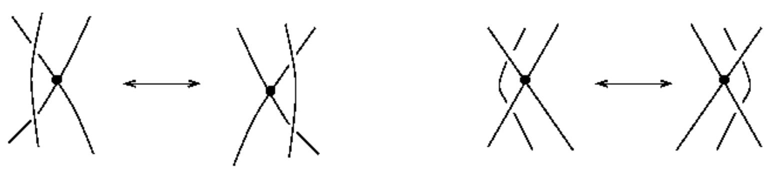


Figure 1.7: Forbidden moves of type $F1$ (on the left) and type $F2$ (on the right)

We can include one or both of them to obtain a "quotient" theory of the theory of virtual links. If we allow the move $F1$, then we obtain the theory of *Welded links* whose interest is growing up recently, in particular because of the fact that the welded braid counterpart can be defined in several equivalent ways. The theory with both forbidden moves added is called the theory of *Fused links* but this theory is trivial, at least at the level of knots, since any knot is equivalent to the trivial knot [**?**, **?**].

Chapter 2

Presentations of surface braid groups

2.1 Introduction

The first presentations of braid groups on closed surfaces were found by Scott ([?]), afterwards revised by Kulikov and Shimada ([?]). Afterwards, González-Meneses provided simpler group presentations for braid groups on closed surfaces. The main result of [?] is a new presentation for braid groups of connected surfaces, possibly with boundary, orientable and non orientable. When Σ is a closed orientable surface, our presentations are similar to González-Meneses' presentations ([?]), but with less relations. At our knowledge, the case of punctured surfaces was new in the literature in the case of oriented surfaces possibly with boundary. Here we recall just the group presentation provided in [?] just in the case of an oriented surface of genus $g \geq 1$ and one boundary component, which will be useful in the following.

Theorem 2.1. **([?])** *Let $\Sigma_{g,1}$ be an oriented surface of genus $g \geq 1$ and with one boundary components. The group $B_n(\Sigma_{g,1})$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, b_1, \dots, b_g$.
- *Relations:*
  - *Braid relations, i.e.*

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\,; \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \textit{for } |i-j| \geq 2\,. \end{aligned}$$

  - *Mixed relations:*

$$\begin{array}{ll} (R1) & a_r\sigma_i = \sigma_i a_r \quad (1 \leq r \leq g;\ i \neq 1)\,; \\ & b_r\sigma_i = \sigma_i b_r \quad (1 \leq r \leq g;\ i \neq 1)\,; \\ (R2) & \sigma_1^{-1}a_r\sigma_1^{-1}a_r = a_r\sigma_1^{-1}a_r\sigma_1^{-1} \quad (1 \leq r \leq g)\,; \\ & \sigma_1^{-1}b_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}b_r\sigma_1^{-1} \quad (1 \leq r \leq g)\,; \\ (R3) & \sigma_1^{-1}a_s\sigma_1 a_r = a_r\sigma_1^{-1}a_s\sigma_1 \quad (s < r)\,; \\ & \sigma_1^{-1}b_s\sigma_1 b_r = b_r\sigma_1^{-1}b_s\sigma_1 \quad (s < r)\,; \\ & \sigma_1^{-1}a_s\sigma_1 b_r = b_r\sigma_1^{-1}a_s\sigma_1 \quad (s < r)\,; \\ & \sigma_1^{-1}b_s\sigma_1 a_r = a_r\sigma_1^{-1}b_s\sigma_1 \quad (s < r)\,; \\ (R4) & \sigma_1^{-1}a_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}a_r\sigma_1 \quad (1 \leq r \leq g)\,. \end{array}$$

Let us also remark that from previous presentation for $B_n(\Sigma_{g,1})$ we can get also a presentation for $B_n(\Sigma_g)$, where $\Sigma - g$ is a closed orientable surface of genus $g \geq 1$, with the same generators and relation plus the additional relation:

$$(TR) \quad [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] = \sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_2\sigma_1\,,$$

where $[a, b] := aba^{-1}b^{-1}$.

In Figure **??** we recall a geometric interpretation of the generators of $B_n(\Sigma_{g,1})$; we represent $\Sigma_{g,1}$ as a polygon with $4g$ sides, equipped with the standard identification of edges and one boundary component. We may consider braids as paths on the polygon, which we draw with the usual 'over and under' information at the crossing points. For the braid a_i (respectively b_j), the only non-trivial path is the first one, which passes through the wall α_i (respectively the wall β_j). The braids $\sigma_1, \ldots, \sigma_{n-1}$ correspond the standard braid generators of B_n, considering $\mathbb{D}_n$ as embedded in $\Sigma_{g,1}$.

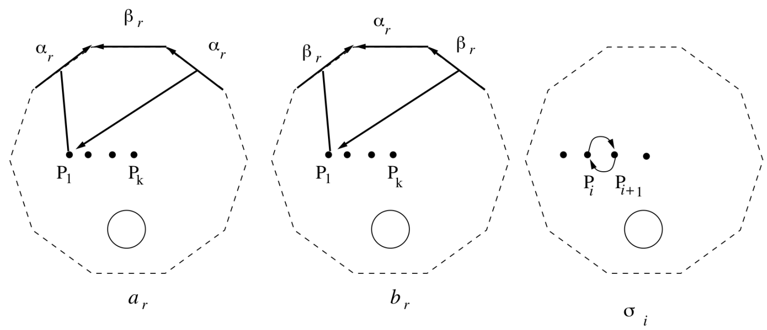


Figure 2.1: The generators $\sigma_1, \ldots, \sigma_{k-1}, a_1, b_1, \ldots, a_g, b_g$

In [**?**] it has been verified that these geometric elements verify all relations provided in previous Theorem: Figure **??** is just a sketch of a homotopy between $\sigma_1^{-1}a_r\sigma_1^{-1}b_r$ and $b_r\sigma_1^{-1}a_r\sigma_1$.

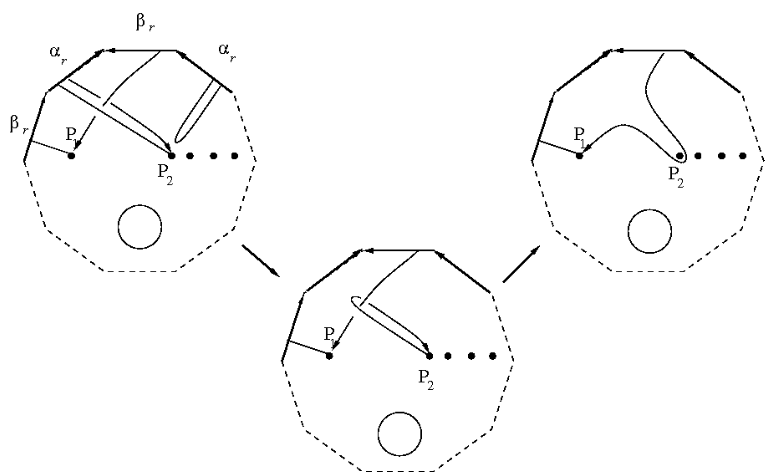


Figure 2.2: Geometric interpretation for relation (R4) in Theorem **??**; homotopy between $\sigma_1^{-1}a_r\sigma_1^{-1}b_r$ (on the left) and $b_r\sigma_1^{-1}a_r\sigma_1$ (on the right).

The proofs in [**?**, **?**, **?**] demand a presentation for the pure subgroup $P_n(\Sigma)$. The proof in [**?**] is different and is inspired by Morita's combinatorial proof for the classical presentation of Artin's braid group ([?]). In [?] we provide also group presentations for pure braid groups on oriented surfaces (possibly with boundary) using a well-known procedure (see for instance [?, ?, ?]) based on Fadell-Neuwith fibration (Chapters 1 and 5) and Theorem 1, Chapter 13 of [?] stating that, given an exact sequence

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1\,,$$

and presentations $\langle G_A, R_A\rangle$ and $\langle G_C, R_C\rangle$, we can deduce a presentation $\langle G_B, R_B\rangle$ for B.

2.2 Positive presentations

The classical presentation of the braid group B_n is positive, i.e. it involves only generators and not their inverses. Hence one can associate with the group B_n a (braid) monoid B_n^+ with the same presentation, but as a monoid presentation. It turns out that the braid monoid B_n^+ is a Garside monoid (see [?]), that is a monoid with a good divisibility structure, and that the braid group B_n is the group of fractions of the monoid B_n^+. As a consequence, the natural morphism of monoids from B_n^+ to B_n is into, and we can solve the word problem, the conjugacy problem and obtain normal forms in B_n (see [?, ?, ?, ?, ?]). These results extend to Artin-Tits groups of spherical type which are algebraic generalization of the braid group B_n.

The main result of [?] is to provide positive presentations for surface braid groups on surfaces of positive genus. This approach was particularly motivated by the fact that questions as the conjugacy problem are not solved in the general case. On the other hand the word problem for surface braid groups is known to be solvable (see [?]) even if algorithms are far to be efficient as the ones proposed for the braid group B_n. In the following we will set $\Sigma_{g,p}$ for a surface of genus g with $p>0$ boundary components and Σ_g for a closed surface of genus g.

Theorem 2.2. ([?]) *Let n and p be positive integers. Let g be a non negative integer. Then, the group $B_n(\Sigma_{g,p})$ admits the following group presentation:*
- *Generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \delta_1, \cdots, \delta_{2g+p-1}$;*
- *Relations:*

-Braid relations:
$(BR1)$ $\sigma_i\,\sigma_j = \sigma_j\,\sigma_i$ *for* $|i-j|\geq 2$;
$(BR2)$ $\sigma_i\,\sigma_j = \sigma_j\,\sigma_i$ *for* $1\leq i\leq n-1$.
- Commutative relations between surface braids:
$(CR1)$ $\delta_r\sigma_i = \sigma_i\delta_r$ *for* $i\neq 1$; $1\leq r\leq 2g+p-1$;
$(CR2)$ $\delta_r\sigma_1\delta_r\sigma_1 = \sigma_1\delta_r\sigma_1\delta_r$ $1\leq r\leq 2g+p-1$;
$(CR3)$ $\delta_r\sigma_1\delta_r\delta_s\sigma_1 = \sigma_1\delta_r\delta_s\sigma_1\delta_r$ *for* $1\leq r<s\leq 2g+p-1$ *with* $(r,s)\neq(p+2i,p+2i+1)$, $0\leq i\leq g-1$.
- Skew commutative relations on the handles:
$(SCR1)$ $\sigma_1\delta_{r+1}\sigma_1\delta_r\sigma_1 = \delta_r\sigma_1\delta_{r+1}$ *for* $r=p+2i$ *where* $0\leq i\leq g-1$.

We remark that the presentation given in Theorem **??** is positive and has less types of relations than the presentation given in Theorem **??** (set $p=1$). In the presentation of Theorem **??**, we can replace Relations $(CR3)$ by the following relations:

$(CR3')$ $\delta_s\sigma_1\delta_r\delta_s\sigma_1 = \sigma_1\delta_r\delta_s\sigma_1\delta_s$

for $1\leq r<s\leq 2g+p-1$ with $(r,s)\neq(p+2i,p+2i+1)$, $0\leq i\leq g-1$. Since the relations of the presentation of $B_n(\Sigma_{g,p})$ are positive, one can define a monoid with the same presentation but as a monoid presentation. It is easy to see that the monoid we obtain does not inject in $B_n(\Sigma_{g,p})$, even if we add the relations of type $(CR3')$ to the presentation given in Theorem **??**. In fact the following relations,

$$(CR3)_k \quad \delta_r\sigma_1\delta_r\delta_s^k\sigma_1 = \sigma_1\delta_r\delta_s^k\sigma_1\delta_r$$

for $1 \leq r < s \leq 2g+p-1$ with $(r,s) \neq (p+2i, p+2i+1)$, $0 \leq i \leq g-1$ and $k \in \mathbb{N}^*$, and

$$(CR3')_k \quad \delta_s \sigma_1 \delta_r^k \delta_s \sigma_1 = \sigma_1 \delta_r^k \delta_s \sigma_1 \delta_s$$

for $1 \leq r < s \leq 2g+p-1$ with $(r,s) \neq (p+2i, p+2i+1)$, $0 \leq i \leq g-1$ and $k \in \mathbb{N}^*$, hold in $B_n(\Sigma_{g,p})$ for each positive integer k, but they are false in the monoid for k greater than 1: no relation of the presentation can be applied to the left side of the equalities. Then starting from the left side of the equality for $k > 1$, we cannot obtain the right side of the equality by using the relations of the monoid presentation only.

Question 2.1. *Let $B_n^*(\Sigma_{g,p})$ be the monoid defined by the presentation of Theorem* **??** *with the extra relations $(CR3)_k$, $(CR3')_k$ for $k \in \mathbb{N}^*$. Is the canonical homomorphism φ from $B_n^*(\Sigma_{g,p})$ to $B_n(\Sigma_{g,p})$ into ?*

In [**?**] we get also similar positive presentations for braid groups of closed surfaces. In the particular case of the group $B_2(\Sigma_1)$ we get positive presentations such that:

- The monoid $B_2^+(\Sigma_1)$ is cancellative (recall that a monoid M is cancellative if the property "$\forall x, y, z, t \in M, (xyz = xtz) \Rightarrow (y = t)$" holds in M) and the canonical morphism $\iota : B_2^+(\Sigma_{1,0}) \longrightarrow B_2(\Sigma_1)$ is into.
- The word and conjugacy problem in $B_2(\Sigma_1)$ are solvable.

The general case appears still out of reach, as well as the study of possible *Garside structures* (see [**?**] for a definition): at this moment, the better approach to relate surface braid groups to Artin-Tits groups and more generally Garside groups is probably to adapt to surface braid groups the group presentations of punctured mapping class groups as quotient of Artin-Tits groups provided in [**?**].

2.3 Group presentations by graphs

Sergiescu [**?**] showed how to associate to any planar, connected graph with n vertices, without loops or intersections, a presentation for the braid group B_n. To each edge e of the graph he associates the braid β_e which is a clockwise half-twist along e (see Figure **??**). Sergiescu provided a complete set of relations using this set of generators for B_n. Afterwards, Birman, Ko and Lee [**?**] extended this result to inner-complete graphs in order to give a new proof for the conjugation problem in B_n. Their group presentation, usually called Birman-Ko-Lee presentation turned out to be useful in other related contexts (see for instance [**?**] and [**?**]), in particular in Garside theory; in fact when we consider Birman-Ko-Lee presentation as a monoid presentation, it coincides with the notion of dual braid monoid (of type $\mathcal{A}$) introduced in [**?**]. Studying the problem of the embedding of monoidal presentations for braid groups in their groups of fractions, Han and Ko [**?**] showed that it is possible to associate braid group presentations to a more general family of graphs (*linearly spanned graphs*) containing the above graphs.

2.3.1 Group presentations for braids on the sphere via graphs

Let us consider now a graph Γ on the sphere $\mathbb{S}^2$ we will say that Γ is *normal* if Γ is connected, finite and it has no loops or intersections. In the following Γ will denote a normal graph on $\mathbb{S}^2$ and $S(\Gamma)$ the set of vertices of Γ. We can associate to the edges of Γ the geometric braids on $\mathbb{S}^2$ as in Figure **??**.

We need some preliminary definitions. Suppose that Γ is not a tree. The set $\mathbb{S}^2 \setminus \Gamma$ is the disjoint union of a finite number of open disks $D_1, \ldots, D_m$, $m > 1$. The boundary of D_j on $\mathbb{S}^2$

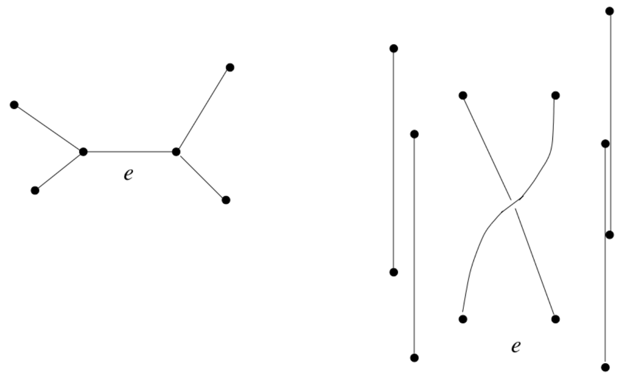


Figure 2.3: Edges and geometric braids.

is a subgraph $\Gamma(D_j)$ of Γ. We choose a point O in the interior of D_j, and an edge σ of $\Gamma(D_j)$ with vertices v_1, v_2. We suppose that the triangle Ov_1v_2 is oriented anticlockwise. We denote σ by $\sigma(e_1)$. We define the *pseudocycle associated to* D_j to be the sequence of edges $\sigma(e_1)\ldots\sigma(e_p)$ such that:

-if the vertex v_{j+1} is not uni-valent, then $\sigma(e_{j+1})$ is the first edge on the left of $\sigma(e_j)$ (we consider $\sigma(e_j)$ going from v_j to v_{j+1}) and the vertex v_{j+2} is the other vertex adjacent to $\sigma(e_{j+1})$;

-if the vertex v_{j+1} is uni-valent, then $\sigma(e_{j+1}) = \sigma(e_j)$ and $v_{j+2} = v_j$.

-the vertex v_{p+1} is the vertex v_1.

Let $\gamma = \sigma(e_1)\ldots\sigma(e_p)$ be a pseudocycle of Γ. Let $i = 1,\ldots,p$. If $\sigma(e_i) = \sigma(e_j)$ for some $j \neq i$, then we say that

- $\sigma(e_i)$ is the *start edge* of a reverse if $j = i+1$ (we set $e_{p+1} = e_1$).

- $\sigma(e_i)$ is the *end edge* of a reverse if $j = i-1$ (we set $e_0 = e_p$).

In the following we set $\sigma_1\ldots\sigma_p$ for the pseudocycle $\sigma(e_1)\ldots\sigma(e_p)$.

Let Δ be a maximal tree of a normal graph Γ on $q+1$ vertices. Then Δ has q edges. Let v_1, v_2 be two vertices adjacent to the same edge σ of Δ. Write $\sigma(f_1)$ for σ. We define the *circuit* $\sigma(f_1)\ldots\sigma(f_{2q})$ as follows:

-if the vertex v_{j+1} is not uni-valent, then $\sigma(f_{j+1})$ is the first edge on the left of $\sigma(f_j)$ (we consider $\sigma(f_j)$ going from v_j to v_{j+1}) and the vertex v_{j+2} is the other vertex adjacent to $\sigma(f_{j+1})$;

-if the vertex v_{j+1} is uni-valent, then $\sigma(f_{j+1}) = \sigma(f_j)$ and $v_{j+2} = v_j$.

This way we come back to v_1 after passing twice through each edge of Δ. Write $\delta_{v_1,v_2}(\Delta)$ for the word in X_Γ corresponding to the circuit $\sigma(f_1)\ldots\sigma(f_{2q})$ (Figure **??**).

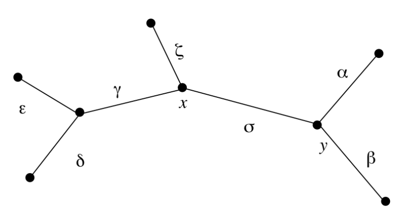


Figure 2.4: $\delta_{x,y}(\Delta) = \sigma\alpha^2\beta^2\sigma\gamma\delta^2\varepsilon^2\gamma\zeta^2$ and $\delta_{y,x}(\Delta) = \sigma\gamma\delta^2\varepsilon^2\gamma\zeta^2\sigma\alpha^2\beta^2$.

Theorem 2.3. **([?])** *Let Γ be a normal graph with n vertices. The braid group $B_n(S^2)$ admits a presentation $\langle X_\Gamma \mid R_\Gamma \rangle$, where $X_\Gamma = \{\sigma \mid \sigma$ is an edge of $\Gamma\}$ and R_Γ is the set of following relations:*

- *Disjointedness relations (DR): if σ_i and σ_j are disjoint, then $\sigma_i\sigma_j = \sigma_j\sigma_i$;*
- *Adjacency relations (AR): if σ_i, σ_j have a common vertex, then $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$;*
- *Nodal relations (NR): if $\{\sigma_1, \sigma_2, \sigma_3\}$ have only one common vertex and they are clockwise oriented (Figure **??**), then*
$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2\,;$$
- *Pseudocycle relations (PR): if $\sigma_1 \dots \sigma_m$ is a pseudocycle and σ_1 is not the start edge or σ_m the end edge of a reverse (Figure **??**), then*
$$\sigma_1\sigma_2\cdots\sigma_{m-1} = \sigma_2\sigma_3\cdots\sigma_m\,.$$
- *Tree relations (TR): $\delta_{x,y}(\Delta) = 1$, for every maximal tree Δ of Γ and every ordered pair of vertices x, y such that they are adjacent to the same edge σ of Δ.*

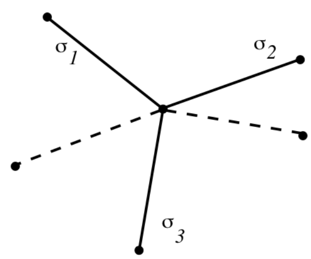


Figure 2.5: Nodal relation.

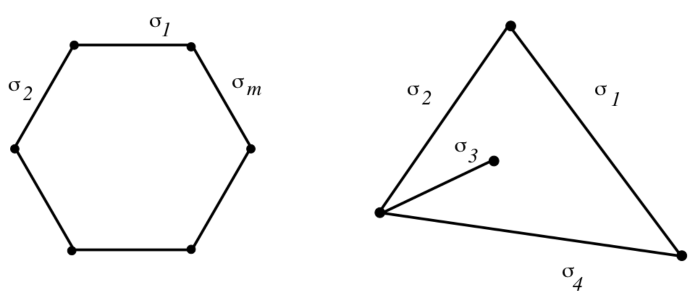


Figure 2.6: pseudocycle relation; on the left $\sigma_1\sigma_2\cdots\sigma_{m-1} = \sigma_2\cdots\sigma_m = \cdots = \sigma_m\cdots\sigma_{m-2}$. On the right $\sigma_1\sigma_2\sigma_3^2 = \sigma_2\sigma_3^2\sigma_4 = \sigma_3^2\sigma_4\sigma_1$ and $\sigma_3\sigma_4\sigma_1\sigma_2 = \sigma_4\sigma_1\sigma_2\sigma_3$.

Remark 2.1. *The statement of Theorem **??** is highly redundant. For instance one can show that a relation (TR) on a given maximal tree of Γ, together with the relations (DR), (AR), (NR) and (PR), generate the (TR) relation for any other maximal tree of Γ. Anyway, these presentations are symmetric and one can read off the relations from the geometry of Γ.*

Corollary 2.1. **([?])** *Every finite group H of the orthogonal group $O(3)$ is isomorphic to a subgroup of $Aut(B_n(S^2))$, where n is the number of vertices of a polyhedron whose symmetry group is H.*

2.3.2 Other surfaces

Let us consider now a graph Γ on an orientable surface Σ. As in the case of the sphere, the graph Γ will be called *normal* if Γ is connected, finite and it has no loops or intersections. Let Γ be a normal graph on Σ and $S(\Gamma)$ the set of vertices of Γ. As before we associate to the edges of Γ the geometric braids on Σ as in Figure **??** and we define $B_\Gamma(\Sigma)$ as the subgroup of $B_{|S(\Gamma)|}(\Sigma)$ generated by these braids. In particular we have the easy following proposition.

Proposition 2.1. ([?]) *Let Σ be an oriented surface such that $\pi_1(\Sigma) \neq 1$ and let Γ be a normal graph on Σ. Then $B_\Gamma(\Sigma)$ is a proper subgroup of $B_{|S(\Gamma)|}(\Sigma)$.*

In the general case we need to introduce different edges corresponding to the fundamental group of the surface Σ: in [**?**] we introduce planar graphs with colored edges and we show how to find presentations via graphs for the braid group on the annulus, $B_n(\Sigma_{0,2})$.

In [**?**] we provide also presentations by graphs for singular braids on the disk and on the annulus, but the case of braids on surfaces of positive genus remains open.

Chapter 3

Finite type invariant for braided objects

The theory of finite type invariants for knots has been widely developed in the 1990. Probably the best paper to introduce this theory and relations with Lie Algebras is [?]. More recently the theory of finite type invariants has been generalized to surface braids [?, ?, ?, ?], virtual knots [?], virtual braids [?], welded braids [?]; a general approach to finite type invariants for virtual and welded knotted objects (tangles, links and braids) is proposed in [?] where finite type invariant theory for "classical" knots appears as a particular case of a more general theory relating Knot theory to the theory of Lie algebras and bialgebras.

3.1 Singular braids on surfaces

Before dealing with finite type invariants let us remind the definition of singular braids, which are related to finite type invariants for braids (and knots). Singular braids have been introduced in [?, ?] as a generalization of classical braids, by allowing strands to intersect in finitely many double points. The set of singular braids on n strands, up to isotopy, forms a monoid usually denoted by SB_n. To the standard generator σ_j of the braid group B_n one can associate the singular braid τ_j, obtained replacing the positive crossing between the j-th and $j+1$-th strands by a singular crossing. To the usual braid relations (and the invertibility of σ_j) one has to add following relations:

- $\tau_i\,\sigma_j\,\sigma_i = \sigma_j\,\sigma_i\,\tau_j$ for $|i-j| = 1$;
- $\tau_i\,\tau_j = \tau_j\,\tau_i$ for $|i-j| \geq 2$;
- $\tau_i\,\sigma_j = \sigma_j\,\tau_i$ for $|i-j| \neq 1$;

Several properties of singular braids have been studied (see for instance [?, ?, ?, ?]) and using generalizations of Hecke algebras [?, ?] it has been constructed a "universal HOMFLY-type invariant" for singular links [?]. In particular, the following results are known.

Theorem 3.1. (Theorem 23 of [?]). *The word problem for SB_n is solvable.*

Theorem 3.2. (Theorems 2.2 and 7.1 of [?]). *For all x in SB_n the following properties are equivalent: (i) $\sigma_j x = x\sigma_k$, (ii) $\sigma_j^r x = x\sigma_k^r$, for some nonzero integer r, (iii) $\sigma_j^r x = x\sigma_k^r$, for any integer r, (iv) $\tau_j x = x\tau_k$, (v) $\tau_j^r x = x\tau_k^r$, for some positive integer r.*

We remark that the proof of Theorem **??** in [?] is combinatorial, whereas the one of Theorem **??** in [?] is geometric and it is related to the notion of *ribbon* (see Definition **??**). We can reformulate Theorem **??** in terms of quasi-centralizers. More precisely, given a (finite) set S of elements in a groups G and an element $s \in S$ we define quasi-centralizer of s an element of $g \in G$ such that $g^{-1}sg = t$ where $t \in S$. Theorem **??** provides therefore a geometric characterization of quasi-centralizers of $\sigma_1, \ldots, \sigma_{n-1}$ in B_n and more generally in the singular braid monoid SB_n in terms of (singular) ribbons. This characterization (now usually called *FRZ property*) has been in particular used by Paris in [?] to solve the so-called *Birman conjecture* for classical braids (see next Section).

Let Σ be a compact, connected and orientable surface. As in the case of classical braids, one can extend the group $B_n(\Sigma)$ to the monoid of singular braids on n strands on Σ allowing a finite number of double points between strands. We denote this monoid $SB_n(\Sigma)$. The monoid $SB_n(\Sigma)$ has been introduced in [?], in order to define finite type invariants for surface braids. A system of generators for $SB_n(\Sigma)$ is provided by a set of generators for $B_n(\Sigma)$, their inverses, and the singular generators $\tau_1, \ldots, \tau_{n-1}$, corresponding to the singular braid generators of SB_n ([?]). Moreover we have that:

Theorem 3.3. **([?])** *The monoid $SB_n(\Sigma)$ embeds in a group.*

The proof of Theorem **??** given in [?] is purely topological and does not use a monoid presentation for $SB_n(\Sigma)$. A different and independent proof of this result is given in my PhD thesis ([?], Theorem 3.4.1). Monoid presentations for $SB_n(\Sigma)$ are given in [?] and [?] (Theorem 3.6.1). We remark also that methods used in [?] for solving the word problem in singular Artin monoids cannot be applied to the presentations of $SB_n(\Sigma)$ provided in [?] and [?], since there are non-homogeneous relations.

The aim of [?] was to extend Theorems **??** and **??** to singular braids on surfaces of positive genus.

Theorem 3.4. **([?])** *Let Σ be an oriented surface of genus $g > 1$. The word problem for $SB_n(\Sigma)$ is solvable.*

Theorem 3.5. **([?])** *Let Σ be an oriented surface of genus $g > 1$. For all x in $SB_n(\Sigma)$, the following properties are equivalent: (i) $\sigma_j x = x\sigma_k$, (ii) $\sigma_j^r x = x\sigma_k^r$, for some nonzero integer r, (iii) $\sigma_j^r x = x\sigma_k^r$, for any integer r, (iv) $\tau_j x = x\tau_k$, (v) $\tau_j^r x = x\tau_k^r$, for some positive integer r.*

Remark 3.1. *Theorem **??** has been used in [?] to prove the Birman conjecture in the case of braids on closed surfaces.*

The key ingredient for proving Theorem **??** is the generalization of notion of ribbon introduced in [?] for classical braids. First let us remind (see Chapter 1) that braids on a closed surface Σ can be considered as mapping classes of the nth punctured surface Σ (when Σ has genus at least 2). In particular let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points in the interior of Σ fixed setwise by any element of $B_n(\Sigma)$.

Definition 3.1. *An arc is an embedding $a : [0,1] \longrightarrow \Sigma$ such that $a(0), a(1)$ are in $\mathcal{P}$ and $a(x)$ is in $\Sigma \setminus \mathcal{P}$ for all x in $(0,1)$. A (j,k)-arc is an arc such that $a(0) = p_j$ and $a(1) = p_k$.*

In particular we identify the braid σ_i with a braid twist permuting p_i and p_{i+1} and we call $[i, i+1]$ an embedded arc associated to σ_i as in Figure **??** (well defined up to isotopy between (j,k)-arcs).

In the following we adopt the convention that, for any β in $B_n(\Sigma)$, $*\beta : \Sigma \longrightarrow \Sigma$ corresponds to a mapping $\Sigma \times \{0\} \longrightarrow \Sigma \times \{1\}$ and defines an action on the right. In particular braids act on the right on the set of arcs on Σ up to isotopy in $\mathrm{Diff}^+(\Sigma, \mathcal{P})$. By abuse of notation, an arc in $\Sigma \times \{0\}$ (respectively in $\Sigma \times \{1\}$) will be an embedding $a : [0,1] \longrightarrow \Sigma \times \{0\}$ (respectively $a : [0,1] \longrightarrow \Sigma \times \{1\}$) as in Definition **??**.

Definition 3.2. *According to [?] we define a* ribbon *as an embedding*

$$R : [0,1] \times [0,1] \longrightarrow \Sigma \times [0,1]\,,$$

such that $R(s,t)$ is in $\Sigma \times \{t\}$. Let β be a braid and let A be a (j,k)-arc in $\Sigma \times \{0\}$. Then the isotopy corresponding to β moves A through a ribbon which is proper *for β, meaning that:*

- *$R(0,t)$ and $R(1,t)$ trace out the strands j and k of the braid β, while the rest of the ribbon is disjoint from β;*
- *$R([0,1] \times \{0\}) = A$ and $R([0,1] \times \{1\}) = A * \beta$.*

We say that the braid β in $B_n(\Sigma)$ has a (j,k)-band if there exists a ribbon proper for β and connecting $[j,j+1] \times \{0\}$ to $[k,k+1] \times \{1\}$.

The following Theorem provides a geometric characterization for quasi-centralizers of $\sigma_1, \ldots, \sigma_{n-1}$

Theorem 3.6. **([?])** *For each β in $B_n(\Sigma)$, the following properties are equivalent:*

1. $\sigma_j \beta = \beta \sigma_k$,
2. $\sigma_j^r \beta = \beta \sigma_k^r$, *for any integer r,*
3. $\sigma_j^r \beta = \beta \sigma_k^r$, *for some nonzero integer r,*
4. *β has a $(j,\ k)$-band,*

Theorem **??** is the key ingredient for proving Theorem **??**: the proof of Theorem **??** is essentially an adaptation of Theorem 2.2 of [?], the main difference being the utilization of a result from [?] computing the intersection number of a curve a and its image under the nth power of a Dehn twist along another curve b (Proposition 3.3 of [?]). Remark also that it is possible to give an algebraic definition of ribbon (see [?]) and Theorem **??** can be generalised also in the case of Artin monoids of spherical type (Proposition 4.1 of [?]).

Finally Theorem **??** can be deduced from Theorem **??** through the following reduction Lemma. First, let $G_{B_n(\Sigma)}$ be the set of generators of $B_n(\Sigma)$ provided in Theorem **??** and $G^{-1}_{B_n(\Sigma)}$ be the set of their inverses. Let T be the set of singular generators $\{\tau_1, \ldots, \tau_{n-1}\}$. Given a word A in $(G^{\pm 1}_{B_n(\Sigma)} \cup T)^*$, we will denote $A(s^+_m)$ and $A(s^-_m)$ the words obtained replacing the m-th singular generator in A by the corresponding positive and negative crossing, respectively. We will write $A(s^e_m)$ for the word obtained replacing the m-th singular generator in A by the identity.

Lemma 3.1. **([?])** *Consider two words A, B in $(G^{\pm 1}_{B_n(\Sigma)} \cup T)^*$. The words A and B represent the same element in $SB_n(\Sigma)$ if and only if there exists $k \in \{1, \ldots, m\}$ such that $A(s^+_1) = B(s^+_k)$ and $A(s^-_1) = B(s^-_k)$ in $SB_n(\Sigma)$.*

A consequence of Lemma **??** is that we can reduce the word problem for $SB_n(\Sigma)$ to the word problem for $B_n(\Sigma)$ which is known to be solvable (see Section 6 of [?]).

3.2 Around the notion of FTI universal invariants for surface braids

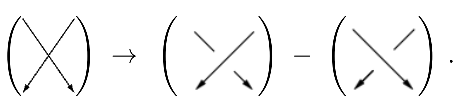
Let us consider a category of embedded 1-dimensional objects like braids, links, tangles etc. There is a natural filtration on the free $\mathbb{Z}$-module generated by the objects, coming from the singular objects with a given number of double points. The main feature of this filtration is that the associated grading, which is usually called the *diagrams algebra*, can be explicitly computed, and has some salient finiteness properties. By universal finite type invariant one generally means a *map* from our category into some completion of the diagrams algebra, which induces an isomorphism at graded level. For instance the celebrated Kontsevich integral, based on the existence of a Drinfel'd associator with rational coefficients, is such a universal invariant. A universal invariant is multiplicative if such map is a group homomorphism. The Kontsevich integral restricted to braids (see [?]) is a universal multiplicative invariant over $\mathbb{Q}$. There exists also a universal invariant for usual braids over $\mathbb{Z}$ ([?]), but it is not known whether there exists a multiplicative one over $\mathbb{Z}$.

In the case of braids and their generalizations the theory of finite type invariants is strictly related to the structure of pure subgroups. Before stating tha main result of [?] we recall a possible definition of finite type invariants for classical braids, which is the algebraical version of the usual definition via singular braids (see [?] or Section 1.3 and Proposition 2.1 of [?] in the case of surface braids). In the following A will denote an abelian group. An invariant of braids is a set mapping $v : B_n \longrightarrow A$. Any invariant $v : B_n \longrightarrow A$ extends by linearity to a morphism of $\mathbb{Z}$-modules $v : \mathbb{Z}[B_n] \longrightarrow A$.

Now, let V be the two-sided ideal of $\mathbb{Z}[B_n]$ generated by $\{\sigma_i - \sigma_i^{-1}, | \, i = 1, \ldots, n-1\}$ and let V^d the d-th power of V. We obtain this way a filtration $\mathbb{Z}[B_n] \supset V \supset V^2 \supset \ldots$ called *Goussarov-Vassiliev filtration* for B_n. A *finite type (Goussarov-Vassiliev) invariant of degree* d is a morphism of $\mathbb{Z}$-modules $v : \mathbb{Z}[B_n] \longrightarrow A$ which vanishes on V^{d+1}.

An universal finite type invariant for B_n is a map from B_n to the completion of the Lie algebra associated to the Goussarov-Vassiliev filtration (according to the different generalizations of B_n this algebra is called *chord diagram algebra* or *arrow diagram algebra*), which induces an isomorphism at the graded level.

In a similar manner, González-Meneses and Paris constructed a universal invariant for braids on surfaces of positive genus (the case of the sphere has been later treated in [?]) using the filtration associated to the desingularization map from $SB_n(\Sigma)$ to $\mathbb{Z}[B_n(\Sigma)]$. More precisely we define a morphism $\partial_n : SB_n(\Sigma) \longrightarrow \mathbb{Z}[B_n(\Sigma)]$ associating to each singular surface braid a linear combination of braids by desingularizing each crossing as follows:



It is quite easy to verify that $\ker \partial_n$ is trivial (the case of B_n was proved in [?]). Since we are dealing with monoids, the triviality of the kernel is not equivalent to the injectivity of ∂_n. The injectivity of ∂_n, known as *Birman conjecture*, was afterwards proved by Paris in the case of classical braids in [?] and for surface braids in [?].

We denote by $\mathcal{V}^d$ the submodule of $\mathbb{Z}[B_n(\Sigma)]$ generated by the desingularisation of (singular) surface braids with d double points. We obtain a filtration $\mathbb{Z}[B_n(\Sigma)] = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \cdots$, that we will call Vassiliev filtration for surface braids. Set $gr_{\mathcal{V}}\mathbb{Z}[B_n(\Sigma)]$ for the associated graded algebra.

González-Meneses and Paris constructed a universal invariant for braids on surfaces $Z : B_n(\Sigma) \longrightarrow gr_{\mathcal{V}}\mathbb{Z}[B_n(\Sigma)]$ but their invariant is not multiplicative. The main result of [?] is to show that actually their result cannot be improved.

Theorem 3.7. **([?])** *There does not exist a multiplicative universal finite type invariant for braids on Σ, when Σ is of genus $g \geq 1$.*

Remark 3.2. *The non-existence of the multiplicative invariant is not related to the ground ring.*

Remark 3.3. *In particular there does not exist any universal invariant for tangles in $\Sigma \times I$ which is functorial with respect to the vertical composition of tangles.*

This result allows us to discuss about the notion of finite type invariants. Our result states that there is not a universal multiplicative FTI invariant (or, using Bar Natan notation from [?], a *homorphic expansion*) for the particular notion of finite type invariants considered in [?]: in other terms we can try to replace the chord diagram algebra considered in [?] with another one arising from a different filtration on the ring algebra of $B_n(\Sigma)$.

In [?] we give also a possible alternative, replacing the Vassiliev filtration $\mathcal{V}^*$ by that associated to the augmentation ideal of the pure braid group of $P_n(\Sigma)$ (see also Chapter 5). In this setting one can find a functorial universal invariant (for the new filtration) which takes values in a larger algebra of *symplectic chord diagrams.* One can present this algebra as the quotient of $\mathbb{Q}[A_s^k, B_t^k, Z_{\alpha i}, Z_{ij}]_{1\leq s,t\leq g, 1\leq\alpha\leq p, 1\leq i,j,k\leq n}$, where deg $A_s^k =$ deg $B_t^k = 1$ and deg $Z_{mj} = 2$ (unless the case when $g = 0$ and one can renormalize the degree of the generators A_{ij} to be 1) by the following relations:

- The extended infinitesimal braid relations:

$$Z_{ij} = Z_{ji}, [Z_{ij}, Z_{kl}] = 0, \text{ if } \{i,j\}\cap\{k,l\} = \emptyset, [Z_{ij}, Z_{jk} + Z_{ik}] = 0, \text{ if } i,j,k \text{ are distinct.}$$

$$[Z_{\alpha j}, Z_{kl}] = 0, \text{ if } j \notin \{k,l\}, [Z_{\alpha j}, Z_{\beta k}] = 0, \text{ if } \{\alpha, j\}\cap\{\beta,k\} = \emptyset, [Z_{\alpha j}, Z_{\alpha k} + Z_{jk}] = 0, \text{ if } j \neq k.$$

- The relations coming from the fundamental group of Σ:

$$[A_s^i, A_r^k] = [B_s^i, B_r^k] = 0, \text{ if } i \neq k, [A_s^i, B_r^j] = 0, \text{ if } r \neq s \text{ and } i \neq j,$$

$$\sum_{s=1}^{g}[A_s^k, B_s^k] + \sum_{j=1}^{n} Z_{jk} + \sum_{\alpha=n+1}^{n+p} Z_{\alpha k} = 0.$$

- The mixed relations:

$$[Z_{jk}, A_s^i] = 0, \text{ if } i \notin \{j,k\}, [Z_{\alpha k}, A_s^i] = 0, \text{ if } i \neq k, [A_s^j + A_s^k, Z_{jk}] = 0, [B_s^j + B_s^k, Z_{jk}] = 0,$$

- The twist relation (making sense only for $g \geq 1$):

$$[A_s^i, B_s^j] = Z_{ij}, \text{ if } i \neq j.$$

The previous presentation turns out to be a presentation for the graded Lie algebra associated to augmentation ideal of $P_n(\Sigma)$. The closed case was proved in [?], the case with boundary is similar. More recently Enriquez and Vershinin proved that the presentations provided in [?] in the case of closed surfaces hold also with integer coefficients.

3.3 FTI for virtual braids

As in the case of braids, the notion of finite type invariants for virtual braids is related to the notion of "pure" subgroups. Define the map $\nu : VB_n \longrightarrow S_n$ of VB_n onto the symmetric group S_n as follows:

$$\nu(\sigma_i) = \nu(\rho_i) = \rho_i, \quad i = 1, 2, \ldots, n-1,$$

where S_n is generated by ρ_i for $i = 1, 2, \ldots, n-1$. The kernel $\ker \nu$ is called *virtual pure braid group on n strands* and it is denoted by VP_n. Using Reidemeister - Schreier method one can verify that the following elements

$$\lambda_{i,i+1} = \rho_i\,\sigma_i^{-1}, \quad \lambda_{i+1,i} = \rho_i\,\lambda_{i,i+1}\,\rho_i = \sigma_i^{-1}\,\rho_i, \quad i = 1, 2, \ldots, n-1,$$

$$\lambda_{i,j} = \rho_{j-1}\,\rho_{j-2}\ldots\rho_{i+1}\,\lambda_{i,i+1}\,\rho_{i+1}\ldots\rho_{j-2}\,\rho_{j-1},$$

$$\lambda_{j,i} = \rho_{j-1}\,\rho_{j-2}\ldots\rho_{i+1}\,\lambda_{i+1,i}\,\rho_{i+1}\ldots\rho_{j-2}\,\rho_{j-1}, \quad 1 \leq i < j-1 \leq n-1.$$

generate VP_n. Moreover we have the following group presentation.

Theorem 3.8. **([?])** *The group VP_n admits a presentation with the generators $\lambda_{k,l}$, $1 \leq k \neq l \leq n$, and the defining relations:*

$$\lambda_{i,j}\,\lambda_{k,l} = \lambda_{k,l}\,\lambda_{i,j}\,; \tag{3.1}$$

$$\lambda_{k,i}\,\lambda_{k,j}\,\lambda_{i,j} = \lambda_{i,j}\,\lambda_{k,j}\,\lambda_{k,i}\,, \tag{3.2}$$

where distinct letters stand for distinct indices.

Remark 3.4. *It is worth remarking that the group VP_n has been independently defined and studied in [?] in relation to Yang Baxter equations. More precisely, according to [?], the virtual pure braid group on n strands is called the n-th quasitriangular group QTr_n and it is generated by $R_{i,j}$ with $1 \leq i \neq j \leq n$ with defining relations given by the Yang Baxter equations*

$$\begin{aligned} R_{i,j}\,R_{k,l} &= R_{k,l}\,R_{i,j}\,; \\ R_{k,i}\,R_{k,j}\,R_{i,j} &= R_{i,j}\,R_{k,j}\,R_{k,i}\,. \end{aligned}$$

Since the natural section $\vartheta : S_n \longrightarrow VB_n$ is well defined one deduces that $VB_n = VP_n \rtimes S_n$. Moreover we can characterize the conjugacy action of S_n on VP_n. Let $VP_n \rtimes S_n$ be the semidirect product defined by the action of S_n on the set $\{\lambda_{k,l} \mid 1 \leq k \neq l \leq n\}$ by permutation of indices.

Proposition 3.1. **([?])** *The map $\omega : VB_n \longrightarrow VP_n \rtimes S_n$ sending any element v of VB_n into $(v((\vartheta \circ \nu)(v))^{-1}, \nu(v)) \in VP_n \rtimes S_n$ is an isomorphism.*

One of the motivations of the study of combinatorial structure of VP_n in [?] was to define a notion of finite type invariant for virtual braids. This notion is coherent with the notion introduced in [?] for virtual knots. Let J be the two-sided ideal of $\mathbb{Z}[VB_n]$ generated by $\{\sigma_i - \rho_i,\, \sigma_i^{-1} - \rho_i \mid i = 1, \dots, n-1\}$. We obtain this way a filtration

$$\mathbb{Z}[VB_n] \supset J \supset J^2 \supset \dots$$

that we call *Goussarov-Polyak-Viro filtration* for VB_n. We will call Goussarov-Polyak-Viro (GPV) invariant of degree d a morphism of $\mathbb{Z}$-modules $v : \mathbb{Z}[VB_n] \longrightarrow A$ which vanishes on J^{d+1}.

This notion of invariant corresponds to the remark that stating with a virtual knot and replacing finitely many crossings (positive or negative) we eventually get the (virtual) unknot.

On the other hand, one can also remark that a GPV invariant restricted to classical braids is a Goussarov-Vassiliev invariant. In fact, since $(\sigma_i - \rho_i) - (\sigma_i^{-1} - \rho_i) = (\sigma_i - \sigma_i^{-1})$ one deduces that $J^d \supset V^d$ where V^d is the d-th power of V, the two-sided ideal of $\mathbb{Z}[VB_n]$ generated by $\{\sigma_i - \sigma_i^{-1}, \mid i = 1, \dots, n-1\}$.

The Goussarov-Polyak-Viro filtration for VB_n corresponds to the I-adic filtration of VP_n. The map $\omega : VB_n \longrightarrow VP_n \rtimes S_n$ defined in Proposition **??** determines an isomorphism of $\mathbb{Z}$-algebras $\Omega : \mathbb{Z}[VB_n] \longrightarrow \mathbb{Z}[VP_n] \otimes \mathbb{Z}[S_n]$, where $\mathbb{Z}[VP_n] \otimes \mathbb{Z}[S_n]$ carries the natural structure of $\mathbb{Z}$-algebra induced by the semi-direct product $VP_n \rtimes S_n$.

Proposition 3.2. **([?])** *The $\mathbb{Z}$-algebras isomorphism $\Omega : \mathbb{Z}[VB_n] \longrightarrow \mathbb{Z}[VP_n] \otimes \mathbb{Z}[S_n]$ sends isomorphically J^d to $I^d(VP_n) \otimes \mathbb{Z}[S_n]$ for all $d \in \mathbb{N}^*$.*

The notion of finite type invariant for virtual knots is particularly useful: in fact, the main result of [?] is that all finite type invariants of (virtual) knots can be constructed in terms of Gauss diagrams. In the case of classical knots this result simplifies the combinatorics involved in computation.

One can wonder, as in the case of (surface) braids, finite type invariants separate virtual braids: this question is equivalent to ask whether VP_n is residually nilpotent (see [?]). This question remains open for $n \geq 4$ (see [?]). On the other hand using different results from [?] and [?], in

[?] it has been proved that there is not a multiplicative universal finite type invariant for virtual braids.

It is interesting to remark that our definition of finite type invariants for virtual braids extend to welded braids. Berceanu and Papadima showed recently that in the case of welded braids one can easily construct a universal multiplicative invariant [?]. We intend to study of relations between Kontsevich integral for classical braids and Berceanu-Papadima universal invariant for welded braids. This work is particularly motivated by [?] where it is explained how Alekseev-Torossian work on Drinfel'd associators and Kashiwara-Vergne [?] can be re-interpreted as a study of welded knotted objects (in particular of *trivalent welded graphs*).

Chapter 4

Automorphisms of braids, braids as automorphisms

4.1 Automorphisms of (surface) braid groups: a geometrical characterization

It is a well known result of Dyer and Grossman [**?**] that the group of outer automorphisms of B_n is isomorphic to cyclic group C_2. In [**?**] is shown that the outer group of the braid group of the annulus (on $n \geq 3$ strands) is isomorphic to $(\mathbb{Z} \rtimes C_2) \times C_2$. In [**?**] we determined the (outer) automorphism groups of braid groups on closed surfaces. The case of surfaces of strictly positive genus this result is based on the geometrical characterization provided in [**?**].

Let Σ be a closed surface of genus g and $R = \Sigma \setminus \{x_1, \dots, x_n\}$. The mapping class group Γ_g^n is clearly isomorphic to the group of isotopy classes of orientation-preserving homeomorphisms $h : R \longrightarrow R$. We recall also that the extended mapping class group of Σ, denoted by $\mathcal{M}^*(\Sigma)$, is the group of isotopy classes of homeomorphisms $h : \Sigma \longrightarrow \Sigma$ while the group of isotopy classes of homeomorphisms $h : R \longrightarrow R$ is called n-punctured extended mapping class group of Σ, $\mathcal{M}_n^*(\Sigma)$.

In [**?**], Irmak, Ivanov and McCarthy proved that each automorphism of $B_n(\Sigma)$ is geometric, i.e., it is induced by a homeomorphism of R, provided that Σ is a closed oriented surface of genus greater than 1 and $n \neq 2$. We can reformulate this result in the following way.

Theorem 4.1. ([?]) *Let Σ be a closed oriented surface of genus g greater than 1 and $n \neq 2$. For each automorphism $x \in Aut(B_n(\Sigma))$ there exists $\hat{x} \in Inn(\mathcal{M}^*(\Sigma))$ such that $\hat{x}_{|B_n(\Sigma)} = x$.*

Theorem **??** has been later proved and extended to the case of non orientable surfaces in [**?**], with a more algebraic and partially based on the study of Scott's group presentation for pure braid groups of closed surfaces.

The first main result of [**?**] is that the map proposed in Theorem **??** provides a characterization for all automorphisms of the braid group on the surface Σ. More precisely we have the following:

Proposition 4.1. ([?]) *Let Σ be a closed oriented surface of genus g greater than 1 and $n \neq 2$. The group $Aut(B_n(\Sigma))$ is isomorphic to $\mathcal{M}_n^*(\Sigma)$.*

In the proof we use in particular that, under our hypothesis, mapping class groups are centerless [**?**]. In another paper Paris and Rolfsen [**?**] showed that when Σ is a closed oriented surface, the centre of $B_n(\Sigma)$ is trivial if $\Sigma \neq S^2, T^2$. As a Corollary of Proposition **??** and previous result one gets the following:

Proposition 4.2. **([?])** *Let Σ be a closed oriented surface of genus greater than 1 and $n \neq 2$. The group $Out(B_n(\Sigma))$ is isomorphic to $\mathcal{M}^*(\Sigma)$.*

Remark 4.1. *The group $B_1(\Sigma)$, by its own definition, is isomorphic to the fundamental group $\pi_1(\Sigma)$. We recall that the isomorphism between $Out(\pi_1(\Sigma))$ and $\mathcal{M}^*(\Sigma)$ was proved by Nielsen.*

Recently Kida and Yamagata proved a deeper result ([?]), namely that if Σ is a closed surface of genus $g \geq 2$ and $n \geq 2$, any isomorphism between subgroups of $B_n(\Sigma)$ is induced by an extended mapping class and therefore that the *abstract commensurator* of $B_n(\Sigma)$ is naturally isomorphic to the extended mapping class group the surface Σ with n disks removed.

4.2 Automorphisms of braid groups of the sphere

The braid group of the sphere $B_n(\mathbb{S}^2)$ may be considered as a quotient of the braid group B_n. According to [?], the group $B_n(\mathbb{S}^2)$ admits the following group presentation:
• Generators: $\sigma_1, \ldots, \sigma_{n-1}$.
• Relations:

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\,; \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{for } |i-j| \geq 2\,; \\ \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 &= 1\,. \end{aligned}$$

The pure braid group on n strands of the sphere $P_n(\mathbb{S}^2)$ is generated by the set $\{A_{i,j} \mid 1 \leq i < j \leq n\}$, where $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$. A group presentation for $P_n(\mathbb{S}^2)$ can be obtained setting $g = 0$ in the group presentation given in Theorem 5.2 of [?].

The group $B_n(\mathbb{S}^2)$ has torsion (see [?] for a determination of all classes of torsion elements). In particular, the element $U = (\sigma_1 \cdots \sigma_{n-1})^n$ is of order two and $ZB_n(\mathbb{S}^2) = ZP_n(\mathbb{S}^2) = \mathbb{Z}_2$.

Theorem 4.2. **([?])** *The group $Out(B_n(\mathbb{S}^2))$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, for $n \geq 4$.*

The proof is essentially combinatorial and provides possible generators for $Out(B_n(\mathbb{S}^2))$; this allows us to deduce some corollaries such as the fact that the group $P_n(\mathbb{S}^2)$ is a *characteristic* subgroup of $B_n(\mathbb{S}^2)$.

Remark 4.2. *The group $B_3(\mathbb{S}^2)$ is of order 12 and it is isomorphic to the group T, the semi-direct product of $\mathbb{Z}_3$ by $\mathbb{Z}_4$ [?]. It can be verified that $Aut(B_3(\mathbb{S}^2)) = B_3(\mathbb{S}^2)$ and $Inn(B_3(\mathbb{S}^2)) = \mathbb{Z}_6$ and then $Out(B_3(\mathbb{S}^2)) = \mathbb{Z}_2$.*

Remark 4.3. *The only open case remains the torus $\mathbb{T}^2$. At our knowledge, no result is known about $Out(B_n(\mathbb{T}^2))$ for n greater than 1. Remark also that the group $B_2(\mathbb{T}^2)/ZB_2(\mathbb{T}^2)$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ the free product of three copies of the cyclic group of order two (see for instance [?, ?]) and according to [?] we have that $Out(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$ is isomorphic to $(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \rtimes S_3$, where the symmetric group S_3 acts by permutation of coordinates.*

4.3 Braid groups of closed surfaces are hopfian and co-hopfian

Let G be a group. We recall that the group G is called hopfian if every surjective endomorphism of G is an isomorphism. The group G is co-hopfian if every injective endomorphism of G is an isomorphism. In [?] was proven that the mapping class group Γ_0^n is co-hopfian: using this result and the fact $B_n(S^2)$ is isomorphic to the quotient of Γ_0^n by its center we prove in [?] that:

Proposition 4.3. **([?])** *The braid group $B_n(S^2)$ is co-hopfian.*

Recently, using some particular complexes of curves, Kida and Yamagata proved that this result is true also for other closed surfaces:

Theorem 4.3. **([?])** *Braid groups of closed surfaces of strictly positive genus are cohopfian.*

On the other hand, it is well-known that the braid group B_n is hopfian. We outline the proof proposed by Lin (Corollary 1.7 of [?]). We recall that a group G is residually finite if the intersection of all finite index subgroups of G is the identity.

Theorem 4.4. **(Malcev [?])** *Any semidirect product of finitely generated residually finite groups is a residually finite group.*

Free groups are residually finite. We recall that the pure braid group P_n is a semidirect product of finitely generated free group (see Chapter 5) and thus, by Theorem **??**, is residually finite.

Theorem 4.5. **(Sec. 41.44 of [?])** *Any finitely generated residually finite group is hopfian.*

Since every finite index subgroup of P_n is also a finite index subgroup in B_n it follows that B_n is also residually finite and then hopfian. On the other hand pure braid groups of the sphere contain a finite index subgroup which is residually torsion free nilpotent (see [?]) (therefore residually finite [?]) and pure braid groups of the torus are residually torsion free nilpotent (see Chapter 5). Therefore we deduce that braid groups of the sphere and of the torus are hopfian.

Finally, mapping class groups are known to be residually finite [?, ?]. From the characterization of braids as mapping classes (Chapter 1) it follows that braid groups on oriented closed surfaces different from the sphere and the torus are (normal) subgroups of mapping class groups and thus they are residually finite. Then we have the following result.

Proposition 4.4. *Let Σ be an oriented surface. The braid group $B_n(\Sigma)$ is hopfian.*

As a last result on automorphisms of surface braid groups we remark that Fel'shtyn and Troitsky recently proved that braid groups of closed surfaces of genus $g \geq 1$ have R_∞ property, i.e. that any automorphism has infinitely many twisted conjugacy classes [?].

4.4 Representations of surface braid groups

In [?] we provided representations of braid groups of orientable surfaces with boundary as automorphisms of finitely-generated free groups. These representations are a particular case of a family of representations of braid groups of orientable surfaces with boundary that, even if not explicitely stated and computed, can be obtained from certain exact sequences given in [?]. In the following we state the results only in the case of $B_n(\Sigma_{g,1})$, according to the group presentation given in Theorem **??**, but similar results were proven in [?] in the general case $B_n(\Sigma_{g,p})$, $p \geq 1$ and also for braid groups of non orientable surfaces with boundary.

Theorem 4.6. **([?])** *Let $U_{n,g}$ be the free group of rank $n+2g$ generated by $\{\tau_1, \dots \tau_n, w_1, \dots, w_{2g}\}$. The group $B_n(\Sigma_{g,1})$ (with $g > 0$ and $n \geq 2$) acts by conjugation on the free group $U_{n,g}$. Therefore we have a representation $\rho_U : B_n(\Sigma_{g,p}) \longrightarrow Aut(U_{n,g})$ defined algebraically as follows:*
– Generators σ_i, $i = 1, \dots, n-1$ of $B_n(\Sigma_{g,1})$:

$$\sigma_i : \begin{cases} \tau_i \longmapsto \tau_{i+1}^{\tau_i^{-1}} \, ; \\ \tau_{i+1} \longmapsto \tau_i \, ; \\ \tau_l \longmapsto \tau_l, \, l \neq i, i+1, \, ; \\ w_r \longmapsto w_r, \, 1 \leq r \leq 2g \, . \end{cases}$$

– *Generators* c_r, $r = 1, \dots, 2g$ *of* $B_n(\Sigma_{g,1})$*, where* $c_{2k+1} = a_k$ *and* $c_{2k} = b_k$*:*

$$c_r : \begin{cases} \tau_1 \longmapsto \tau_1^{w_r^{-1}\tau_1}\,; \\ \tau_i \longmapsto \tau_i,\ 2 \leq i \leq n\,; \\ w_s \longmapsto w_s^{[w_r^{-1},\tau_1]},\ s < r,\ (s,r) \neq (2m-1, 2m)\,; \\ w_{r-1} \longmapsto [\tau_1, w_r^{-1}]w_{r-1}\tau_1,\ \textit{if}\ r = 2m\,; \\ w_r \longmapsto w_r^{\tau_1}\,; \\ w_s \longmapsto w_s,\ r < s,\ (r,s) \neq (2m-1, 2m)\,; \\ w_{r+1} \longmapsto \tau_1^{-1}w_{r+1},\ \textit{if}\ r = 2m-1\,. \end{cases}$$

One of the main results of [?] is the following:

Theorem 4.7. ([?]) *The representation* $\rho_U : B_n(\Sigma_{g,p}) \longrightarrow \mathrm{Aut}(U_{n,g})$ *is faithful.*

As we recalled in the Introduction, any element β of $B_n \subset \mathrm{Aut}(F_n)$ fixes the product $x_1x_2\dots x_n$ of generators of F_n. We proved a similar statement for the group $B_n(\Sigma_{g,p})$, for $p > 0$. As before we state here only the case $p = 1$.

Proposition 4.5. ([?]) *Any element* β *in* $B_n(\Sigma_{g,1}) \subset \mathrm{Aut}(U_{n,g})$ *fixes the element* $\tau_n^{-1}\cdots\tau_2^{-1}\tau_1^{-1}[w_1^{-1}, w_2]\cdots[w_{2g-1}^{-1}, w_{2g}]$.

Let $p : \mathrm{Aut}(U_{n,g}) \longrightarrow \mathrm{Out}(U_{n,g})$ be the canonical projection.

Theorem 4.8. ([?]) *The representation* $p \circ \rho_U : B_n(\Sigma_{g,1}) \longrightarrow \mathrm{Out}(U_{n,g})$ *is faithful for* $g > 0$.

Remark 4.4. *Let* $\varphi : B_n \longrightarrow \mathrm{Out}(F_n)$ *be the representation obtained by composing Artin's representation of* B_n *in* $\mathrm{Aut}(F_n)$ *with the canonical projection of* $\mathrm{Aut}(F_n)$ *in* $\mathrm{Out}(F_n)$*. This representation is not faithful, and it is easy to see that its kernel is the center of* B_n*.*

On the other hand, contrarily to Artin representation, our representations do not belong to the Torelli group of $U_{n,g}$, the group of automorphisms of $U_{n,g}$ which become trivial in the abelianization. Therefore is not possible to extend Burau representation using Fox derivatives: this obstruction turns out to be deeper, see Propositions 5.4, 5.7 and 5.8 of [?] and Chapter 5.

As already noticed by several authors (see for instance Section 2.2 of [?]), it is not possible to extend Artin representations to braid groups of closed oriented surfaces. In [?] we provide an algebraic argument for this fact and we give a representation of braid groups of closed surfaces in the outer automorphism group of a finitely generated free group.

Question 4.1. *The representation of* $B_n(\Sigma_g)$ *in* $\mathrm{Out}(F_{n+g})$ *given in Proposition 7 of [?] is it faithful?*

4.5 Representations of Artin-Tits and virtual braid groups

We recall that classical braid groups are also called Artin-Tits groups of type $\mathcal{A}$. More precisely, let (W, S) be a Coxeter system and let us denote the order of the element st in W by $m_{s,t}$ (for $s, t \in S$). Let $A(W)$ be the group defined by the following group presentation:

$$A(W) = \langle S \mid \underbrace{st\cdots}_{m_{s,t}} = \underbrace{ts\cdots}_{m_{s,t}} \ \text{ for any } s \neq t \in S \text{ with } m_{s,t} < +\infty \rangle\ .$$

The group $A(W)$ is the Artin-Tits group associated to W. The group $A(W)$ is said to be of spherical type if W is finite. There exist three infinite families of finite Coxeter groups usually denoted respectively as of type $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$. Artin braid groups correspond to the family of Artin-Tits groups associated to Coxeter groups of type $\mathcal{A}$, and braid groups of the annulus coincide with Artin-Tits groups associated to Coxeter groups of type $\mathcal{B}$. Artin Tits of type $\mathcal{B}$ can be easily seen as a geometric subgroup of Artin Tits of type $\mathcal{A}$ and therefore can be represented as automorphisms of finitely-generated free groups. In this Section we give a faithful representation of the remaining infinite family of spherical Artin-Tits groups, associated to Coxeter groups of type $\mathcal{D}$.

First let us denote by $A(\mathcal{D}_n)$ the n-th Artin-Tits group of type $\mathcal{D}$ with the group presentation given by its Coxeter graph (see Figure **??**), where the vertices $\delta_1, \ldots, \delta_n$ are the generators of the group, and two generators δ_i, δ_j satisfy the relation $\delta_i\delta_j\delta_i = \delta_j\delta_i\delta_j$ if they are related by an edge, and commute otherwise.

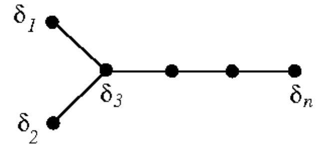


Figure 4.1: Coxeter graph of type $\mathcal{D}$

We denote by $\pi_D : A(\mathcal{D}_n) \longrightarrow B_n$ the epimorphism defined by $\pi_D(\delta_1) = \pi_D(\delta_2) = \sigma_1$ and $\pi_D(\delta_i) = \sigma_{i-1}$ for $i = 3, \ldots, n$, which admits the section $s_D : B_n \longrightarrow A(\mathcal{D}_n)$ defined by $s_D(\sigma_i) = \delta_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Proposition 4.6. *1. The representation $\rho_D : B_n \longrightarrow Aut(F_{n-1})$ given in Chapter 1 is well defined and faithful.*

2. The group $A(\mathcal{D}_n)$ is isomorphic to $F_{n-1} \rtimes_{\rho_D} B_n$, where the projection on the second factor is π_D and the section $B_n \longrightarrow F_{n-1} \rtimes_{\rho_D} B_n$ is just s_D.

3. In particular, $\ker \pi_D = F_{n-1}$ is freely generated by $\lambda_1, \ldots, \lambda_{n-1}$, where $\lambda_1 = \delta_1\delta_2^{-1}$ and $\lambda_i = (\delta_{i+1}\cdots\delta_3)(\delta_1\delta_2^{-1})(\delta_{i+1}\cdots\delta_3)^{-1}$ for $i = 2, \ldots, n-1$.

The first item of Proposition **??** was established in [**?**] by topological means and afterwards in [**?**], and the second and the third items are proven in Proposition 2.3 of [**?**]. From Proposition **??** we deduce in [**?**] a faithful representation of $A(\mathcal{D}_n)$ into $\mathrm{Aut}(F_n)$.

Proposition 4.7. **([?])** *The representation $\iota : A(\mathcal{D}_n) \longrightarrow \mathrm{Aut}(F_n)$ given algebraically by:*

$$\iota(\delta_1) : \begin{cases} x_1 \longmapsto x_1, & \\ x_j \longmapsto x_j x_1^{-1}, & j \neq 1,\, n, \\ x_n \longmapsto x_1 x_n x_1^{-1}, & \end{cases}$$

$$\iota(\delta_2) : \begin{cases} x_1 \longmapsto x_1, & \\ x_j \longmapsto x_1^{-1} x_j, & j \neq 1,\, n, \\ x_n \longmapsto x_n, & \end{cases}$$

and for $3 \leq i \leq n$,

$$\iota(\delta_i) : \begin{cases} x_{i-2} \longmapsto x_{i-1}, & \\ x_{i-1} \longmapsto x_{i-1} x_{i-2}^{-1} x_{i-1}, & \\ x_j \longmapsto x_j, & j \neq i-2, i-1. \end{cases}$$

is well defined and faithful.

In a work in progress with Bardakov and González-Meneses [?] we consider the case affine braid group of type A. We recall that the affine braid group of type $\mathcal{A}$ on n strands, $A(\tilde{A}_{n-1})$, can be defined as the group obtained by the group presentation of B_{n+1} by replacing the relation $\sigma_n\sigma_1 = \sigma_1\sigma_n$ with the relation $\sigma_n\sigma_1\sigma_n = \sigma_1\sigma_n\sigma_1$. It is also known as circular braid group [?] and it can be described geometrically as a finite index subgroup of the mapping class group Γ_0^{n+2} [?]. The map $\iota : A(\tilde{A}_{n-1}) \longrightarrow B_{n+1}$ defined by $\iota(\sigma_j) = \sigma_j$ for $j = 1, \ldots, n-1$ and $\iota(\sigma_n) = \sigma_n^{-2}\sigma_{n-1}^{-1} \cdots \sigma_2^{-1}\sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_n^2$ is well defined and injective (see Section 2 of [?]). Composing with Artin representation we get a faithful representation $\rho_{\tilde{A}} : A(\tilde{A}_{n-1}) \longrightarrow \mathrm{Aut}(F_{n+1})$. In [?] we use this result to find a representation of $A(\tilde{A}_{n-1})$ into $\mathrm{Aut}(F_n)$ that we prove to be faithful using well known relations between braid groups and affine braid groups.

Theorem 4.9. **([?])** *The representation $\rho_{Aff} : A(\tilde{A}_{n-1}) \longrightarrow \mathrm{Aut}F_n$ defined as follows:*

$$i \neq n : \rho_{Aff}(\sigma_i) : \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases}$$

$$\rho_{Aff}(\sigma_n) : \begin{cases} x_1 \longmapsto x_n & \\ x_n \longmapsto x_n x_1 x_n^{-1}, & \\ x_l \longmapsto x_l, & l \neq 1, n. \end{cases}$$

is faithful.

One possible application of Theorem **??** is given by virtual braid groups and their representation as automorphisms of free groups:

Theorem 4.10. **[?]** *There is a representation ψ of VB_n in $Aut(F_{n+1})$, $F_{n+1} = \langle x_1, x_2, \ldots, x_n, y\rangle$ which is defined by the following actions on the generators of VB_n:*

$$\psi(\sigma_i) : \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1; \\ y \longmapsto y, & \end{cases} \qquad \psi(\rho_i) : \begin{cases} x_i \longmapsto y\, x_{i+1}\, y^{-1}, & \\ x_{i+1} \longmapsto y^{-1}\, x_i\, y, & \\ x_l \longmapsto x_l, & l \neq i, i+1, \\ y \longmapsto y, & \end{cases}$$

for all $i = 1, 2, \ldots, n-1$.

The faithfulness of the representation given in Theorem **??** is evident for $n = 2$ while the faithfulness for $n = 3$ is related to Theorem **??** (see [?]). For $n > 3$ we do not know if above representation is faithful. Remark that the group WB_n can be considered as a quotient of $\psi(VB_n)$: in fact a straightforward verification shows that:

Proposition 4.8. *Let $q_n : VB_n \longrightarrow WB_n$ be the projection defined by $q_n(\sigma_i) = \sigma_i$ and $q_n(\rho_i) = \alpha_i$ for $i = 1, \ldots, n$. Let $F_{n+1} = \langle x_1, x_2, \ldots, x_n, y\rangle$ and $F_n = \langle x_1, x_2, \ldots, x_n\rangle$. The projection $p_n : F_{n+1} \longrightarrow F_{n+1}/\langle\langle y\rangle\rangle \simeq F_n$ induces a map $p_n^{\#} : \psi(VB_n) \longrightarrow WB_n$ such that $p_n^{\#} \circ \psi = q_n$.*

Since $\psi(VB_n) \subseteq WB_{n+1}$ the faithfulness of ψ for any n would imply that virtual braid groups can be considered as subgroup of welded groups, whose structure and applications in finite type invariants theory is much more advanced (see Chapter 3 and [?, ?]).

4.6 An application

Theorem **??** was used in [?] for constructing a Burau-like representation for VB_n. In [?] we use Theorem **??** for defining a new notion of group of a virtual knot.

In the classical case the group of a link L is defined as the fundamental group $\pi_1(S^3 \setminus N(L))$ where $N(L)$ is a tubular neighborhood of the link in S^3.

To find a group presentation of this group we can use Wirtinger method. We recall briefly that given an oriented diagram D_L of the link L, $\pi_1(S^3 \setminus N(L))$ admits the following group presentation: generators are loops $\{a_1, \dots a_n\}$, where a_j is the loop associated to the arc A_j of D_L and relations correspond to crossings in D_l as follows:

$$\begin{array}{ll} a_j a_k = a_k a_i & \text{if } a_i, a_j, a_k \text{ meet in a crossing like in case a) of figure } \textbf{??}; \\ a_k a_j = a_i a_k & \text{if } a_i, a_j, a_k \text{ meet in a crossing like in case b) of figure } \textbf{??}. \end{array}$$

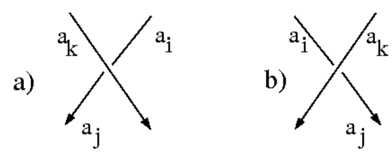


Figure 4.2: Arcs around two types of crossings

We recall that this presentation is usually called upper Wirtinger presentation while the lower Wirtinger presentation is obtained applying Wirtinger method to the diagram where all crossings are reversed; these presentations are generally different but the corresponding groups are evidently isomorphic because of their geometrical meaning.

Another way to obtain a group presentation for $\pi_1(S^3 \setminus N(L))$ is to consider a braid $\beta \in B_n$ such that its Alexander closure is isotopic to L; therefore the group $\pi_1(S^3 \setminus N(L))$ admits the presentation:

$$\pi_1(S^3 \setminus N(L)) = \langle x_1, x_2, \dots, x_n \mid\mid x_i = \beta(x_i), \quad i = 1, \dots, n \rangle ,$$

where we consider β as an automorphism of F_n (this is a consequence of van Kampen's Theorem, [**?**]).

Given a virtual link vL, according to [**?**] the group of the virtual knot vL, denoted with $G_{K,v}(vL)$, is the group obtained extending the Wirtinger method to virtual diagrams, forgetting all virtual crossings. This notion of group of a virtual knot (or link) is not satisfactory: for instance if vT is the virtual trefoil knot with two classical crossings and one virtual crossing and U is unknot then $G_{K,v}(vT) \simeq G_{K,v}(U) \simeq \mathbb{Z}$, although that vT is not equivalent to U.

In [**?**] we introduced another notion of group $G_v(vL)$ of a virtual link vL. Let $vL = \widehat{\beta_v}$ be a closed virtual braid, where $\beta_v \in VB_n$ [1]. Define

$$G_v(vL) = \langle x_1, x_2, \dots, x_n, y \mid x_i = \psi(\beta_v)(x_i), \quad i = 1, \dots, n \rangle.$$

Using Kamada characterization [**?**] of virtual braids having equivalent closures as virtual links we prove in [**?**] that:

Theorem 4.11. **([?])** *The group $G_v(vL)$ is an invariant of the virtual link vL.*

In [**?**] we prove also that this invariant is stronger than Kauffman invariant:

Proposition 4.9. **([?])** *Let vK be a virtual knot.*

1. *The group $G_v(vK)/\langle\langle y \rangle\rangle$ is isomorphic to $G_{K,v}(vK)$;*
2. *The abelianization of $G_v(vK)$ is isomorphic to $\mathbb{Z}^2$.*

[1]The index v is given to precise when we are considering virtual braids

The group $G_v(K)$ is not a complete invariant for virtual knots. Let $c = \rho_1\sigma_1\sigma_2\sigma_1\rho_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} \in VB_3$; the closure $\widehat{c}$ is equivalent to the Kishino knot (see Figure **??**). The Kishino knot is a non trivial virtual knot [**?**] with trivial Jones polynomial and trivial fundamental group ($G_{K,v}(\widehat{c}) = \mathbb{Z}$). For this knot we have that $G_v(\widehat{c}) = F_2$.

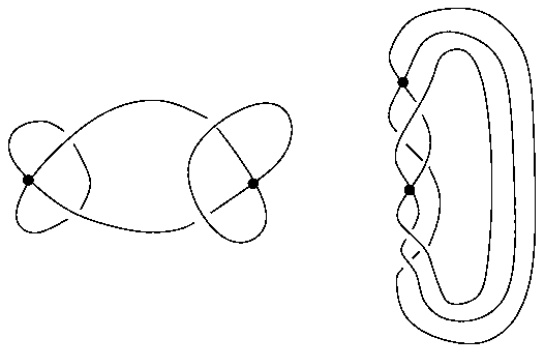


Figure 4.3: The Kishino knot: usual diagram and as the closure of a virtual braid.

However, the notion of group system of a knot can be combinatorially extended to $G_v(K)$ (Remarks 8 and 9 of [**?**]): we do not know if such a group system is or not a complete invariant for virtual knots. Similarly it is possible to define an invariant for welded links using the interpretation of welded braids as automorphisms of F_n: moreover, using an extension of the notion of Wada representations (introduced Chapter 1) to WB_n we obtain in [**?**] new families of invariants for welded links.

With Bardakov and Meilhan we intend to study the notion of Milnor's invariants related to $G_v(K)$ and to compare to the results in [**?**].

Chapter 5

Lower central series: results and applications

5.1 Lower central series for Artin-Tits groups

Given a group G, we recall that the *lower central series* of G is the filtration $G := \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$, where $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ for $i \geq 2$. The group G is said to be *perfect* if $G = \Gamma_2(G)$. From the lower central series of G one can define another filtration $D_1(G) \supseteq D_2(G) \supseteq \cdots$ by setting $D_1(G) = G$, and for $i \geq 2$, defining $D_i(G) = \{\, x \in G \,|\, x^n \in \Gamma_i(G)$ for some $n \in \mathbb{N}^* \,\}$. This filtration was first considered by Stallings [**?**] and we call it the *rational lower central series* of G, as proposed by [**?**].

Following P. Hall, for any group-theoretic property $\mathcal{P}$, a group G is said to be *residually* $\mathcal{P}$ if for any (non-trivial) element $x \in G$, there exists a group H with the property $\mathcal{P}$ and a surjective homomorphism $\varphi : G \longrightarrow H$ such that $\varphi(x) \neq 1$. It is well known that a group G is residually nilpotent if and only if $\bigcap_{i\geq 1} \Gamma_i(G) = \{1\}$. On the other hand, a group G is residually torsion-free nilpotent if and only if $\bigcap_{i\geq 1} D_i(G) = \{1\}$.

Let us start by recalling some standard results on combinatorial properties of braid groups. The following result is well known.

Proposition 5.1. (see for instance [?]) *Let B_n be the Artin braid group on $n \geq 3$ strands. Then $\Gamma_1(B_n)/\Gamma_2(B_n) \cong \mathbb{Z}$ and $\Gamma_2(B_n) = \Gamma_3(B_n)$.*

In [**?**] we prove an analogous result for Artin-Tits groups of spherical type.

Proposition 5.2. ([?]) *Let B_W be an Artin-Tits group of spherical type, where W is different from the dihedral group I_{2m}. Then:*

i) If $m_{s,t}$ is either odd or equal to 2 for any pair s, t in W then $\Gamma_1(B_W)/\Gamma_2(B_W)$ is isomorphic to $\mathbb{Z}$, and is isomorphic to $\mathbb{Z}^2$ otherwise.

ii) $\Gamma_2(B_W) = \Gamma_3(B_W)$.

Remark 5.1. *The Artin-Tits group of type $\mathcal{B}$ with two generators is isomorphic to I_4 and therefore is not included in Proposition* **??**. *The Artin-Tits group $B_{I_{2m}} = \langle a, b \,|\, (ab)^m = (ba)^m \rangle$ is residually nilpotent if and only if m is a power of a prime number. Indeed, by taking $c = ba$, it is readily seen that the group B_W is isomorphic to the Baumslag-Solitar group of type (m, m), $BS_m = \langle a, c \,|\, [a, c^m] = 1 \rangle$, which is known to be residually nilpotent if m is a power of a prime number.*

Conversely, using a result of Moldavanskii on isomorphisms of Baumslag-Solitar groups ([?]) we show also that $B_{I_{2m}}$ is residually nilpotent if and only if m is a power of a prime number.

It is well known that pure braid groups are residually torsion-free nilpotent [?]. Using the faithfulness of the Krammer-Digne representation, Marin has shown that the pure Artin-Tits groups of spherical type are also residually torsion-free nilpotent [?].

The fact that a group is residually torsion-free nilpotent has several important consequences, notably that the group is *bi-orderable* [?]. We recall that a group G is said to be bi-orderable if there exists a strict total ordering $<$ on its elements which is invariant under left and right multiplication, in other words, $g < h$ implies that $gk < hk$ and $kg < kh$ for all $g, h, k \in G$. We state one interesting property of bi-orderable groups. A group G is said to have *generalised torsion* if there exist $g, h_1, \ldots, h_k$, $(g \neq 1)$ such that:

$$(h_1 g h_1^{-1})(h_2 g h_2^{-1}) \cdots (h_k g h_k^{-1}) = 1\,.$$

Proposition 5.3. **([?])** *A bi-orderable group has no generalised torsion.*

The braid group B_n is not bi-orderable for $n \geq 3$ since it has generalised torsion (see [?]).

5.2 Lower central series for surface braid groups

The main results of [?], which concern orientable surfaces of genus at least one, are as follows.

Theorem 5.1. **([?])** *Let Σ_g be a compact, connected orientable surface without boundary, of genus $g \geq 1$, and let $n \geq 3$. Then:*

(a) $\Gamma_1(B_n(\Sigma_g))/\Gamma_2(B_n(\Sigma_g)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$.

(b) $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g)) \cong \mathbb{Z}_{n-1+g}$.

(c) $\Gamma_3(B_n(\Sigma_g)) = \Gamma_4(B_n(\Sigma_g))$. *Moreover* $\Gamma_3(B_n(\Sigma_g))$ *is perfect for* $n \geq 5$.

(d) $B_n(\Sigma_g)$ *is not residually nilpotent.*

This implies that braid groups of compact, connected orientable surfaces without boundary may be distinguished by their lower central series (indeed by the first two lower central quotients).

Theorem 5.2. **([?])** *Let $g \geq 1$, $m \geq 1$ and $n \geq 3$. Let $\Sigma_{g,m}$ be a compact, connected orientable surface of genus g with m boundary components. Then:*

(a) $\Gamma_1(B_n(\Sigma_{g,m}))/\Gamma_2(B_n(\Sigma_{g,m})) = \mathbb{Z}^{2g+m-1} \oplus \mathbb{Z}_2$.

(b) $\Gamma_2(B_n(\Sigma_{g,m}))/\Gamma_3(B_n(\Sigma_{g,m})) = \mathbb{Z}$.

(c) $\Gamma_3(B_n(\Sigma_{g,m})) = \Gamma_4(B_n(\Sigma_{g,m}))$. *Moreover* $\Gamma_3(B_n(\Sigma_{g,m}))$ *is perfect for* $n \geq 5$.

(d) $B_n(\Sigma_{g,m})$ *is not residually nilpotent.*

In [?, ?] we provide also group presentations for $B_n(\Sigma_{g,m})/\Gamma_3(B_n(\Sigma_{g,m}))$.

Corollary 5.1. **([?, ?])** *Let $n \geq 3$ and $g \geq 1$.*

- *The group $B_n(\Sigma_{g,1})/\Gamma_3(B_n(\Sigma_{g,1}))$ is isomorphic to the semi-direct product $(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$. More precisely, according to the presentation in Theorem **??**, the first factor $\mathbb{Z}$ is central and is generated by σ, the second factor $\mathbb{Z}^g$ is generated by $\{a_1, \ldots, a_g\}$, and the third factor $\mathbb{Z}^g$ is generated by $\{b_1, \ldots, b_g\}$. Any generator b_j (for $1 \leq j \leq g$) acts trivially on $a_1, \ldots, a_{j-1}, a_{j+1}$ and $b_j a_j b_j^{-1} = \sigma^{-2} a_j$. Hence, $B_n(\widehat{\Sigma}_g)\,/\Gamma_3(B_n(\widehat{\Sigma}_g))$ is a central extension of $\mathbb{Z}^{2g}$ by $\mathbb{Z}$.*

- *The group* $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ *is isomorphic to the semidirect product* $\left(\mathbb{Z}_{2(n+g-1)} \times \mathbb{Z}^g\right) \rtimes \mathbb{Z}^g$*, with* $2g+1$ *factors respectively generated by* $\sigma, a_1, b_1, \ldots, a_g, b_g$ *as above.*

Since for $n = 1$ we have that $B_1(\Sigma) = \pi_1(\Sigma)$, which is residually free, in order to complete the study of lower central series of braid groups of closed surfaces we need to consider the case of 2 strands. In [**?**] using ideas from [**?**] and results of [**?**], we show that apart from the first term, the lower central series of $B_2(\mathbb{T})$ and $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ coincide, and we also determine all of their successive lower central quotients: we show in particular that $B_2(\mathbb{T})$ is residually nilpotent but not residually torsion-free nilpotent since it has generalised torsion.

The general case of braid groups on 2 strands has been solved in [**?**].

Proposition 5.4. ([?]) *Let* $g \geq 1$*. The group* $B_2(\Sigma_g)$ *is residually* 2*-finite. In particular* $B_2(\Sigma_g)$ *is residually nilpotent.*

The proof proposed in [**?**] follows as a consequence of the fact that surface pure braid groups are residually nilpotent and of a result of Gruenberg on residual properties of extensions ([**?**]).

Question 5.1. *Let* $g \geq 1$*. The group* $B_2(\Sigma_g)$ *is bi-orderable?*

5.3 Fadell-Neuwirth fibrations and almost direct products

Let Σ be an orientable surface.

Regarded as a subgroup of $\mathfrak{S}_{k+n}$, the group $\mathfrak{S}_k \times \mathfrak{S}_n$ acts on $\mathbb{F}_{k+n}(\Sigma)$. The fundamental group $\pi_1\left(\mathbb{F}_{k+n}(\Sigma)/(\mathfrak{S}_k \times \mathfrak{S}_n)\right)$ is called the *mixed braid group of* Σ *on* (k,n) *strands*, and shall be denoted by $B_{k,n}(\Sigma)$. Notice that $B_{k,n}(\Sigma)$ embeds canonically in $B_{k+n}(\Sigma)$. These intermediate groups between pure braid and braid groups of a surface, known as 'mixed' braid groups have been defined in [**?**, **?**, **?**], and studied in more detail in [**?**] when Σ is the 2-sphere $\mathbb{S}^2$.

The map $\mathbb{F}_{k+n}(\Sigma)/(\mathfrak{S}_k \times \mathfrak{S}_n) \longrightarrow \mathbb{F}_n(\Sigma)/\mathfrak{S}_n$ given by forgetting the first k coordinates is a locally-trivial fibration whose fibre may be identified with $\mathbb{F}_k(\Sigma \setminus \{x_1, \ldots, x_n\})/\mathfrak{S}_k$. As in the case of pure braid groups (recalled in Chapter 1), the long exact sequence in homotopy of this fibration yields a short exact sequence.

Lemma 5.1. (see for instance [?]) *Let* $k, n \in \mathbb{N}$*. The Fadell-Neuwirth fibration* $\mathbb{F}_{k+n}(\Sigma) \longrightarrow \mathbb{F}_n(\Sigma)$ *induces the short exact sequence:*

$$1 \longrightarrow B_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \longrightarrow B_{k,n}(\Sigma) \longrightarrow B_n(\Sigma) \longrightarrow 1, \tag{MSB}$$

where we suppose that $n \geq 3$ *if* $\Sigma = \mathbb{S}^2$.

In what follows we shall refer to the above short exact sequence as (**??**) (mixed surface braid groups sequence), and we denote its restriction to the corresponding pure braid groups

$$1 \longrightarrow P_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \longrightarrow P_{k+n}(\Sigma) \longrightarrow P_n(\Sigma) \longrightarrow 1, \tag{SPB}$$

by (**??**) (surface pure braid groups sequence). If Σ is the disc $\mathbb{D}^2$, we recover the sequence usual decomposition of P_n as an iterated semi-direct product of free groups.

We have the following commutative diagram involving the short exact sequences (**??**) and (**??**):

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & P_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \arrow[r] \arrow[d] & P_{k+n}(\Sigma) \arrow[r] \arrow[d] & P_n(\Sigma) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \arrow[r] \arrow[d] & B_{k,n}(\Sigma) \arrow[r] \arrow[d] & B_n(\Sigma) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathfrak{S}_k \arrow[r] \arrow[d] & \mathfrak{S}_k \times \mathfrak{S}_n \arrow[r] \arrow[d] & \mathfrak{S}_n \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{5.1}$$

where the vertical arrows between (**??**) and (**??**) are inclusions, and the second vertical sequence is obtained by sending any element of $B_{k,n}(\Sigma)$ in the associated permutation. The third row of symmetric groups splits as a direct product.

The following Theorem summarises some of the known results for the splitting problem for the sequences (**??**) and (**??**): the main reference is [**?**], except results for the sequence (**??**), that we proved in [**?**]. We refer to [**?**, **?**] for the case of $\mathbb{S}^2$.

Theorem 5.3. **([?])** *Let Σ be a compact, connected orientable surface different from $\mathbb{S}^2$.*

(a) Suppose that Σ has empty boundary.

- *(i) If Σ is the 2-torus $\mathbb{T}^2$ then (**??**) splits for all $k, n \in \mathbb{N}$.*
- *(ii) If $n = 1$ then both (**??**) and (**??**) split for all $k \in \mathbb{N}$.*
- *(iii) Let $n \geq 2$ and $k \in \mathbb{N}$. If $\Sigma \neq \mathbb{T}^2$, then (**??**) does not split. If further $k = 1$ then (**??**) does not split.*

*(b) If Σ has non-empty boundary then (**??**) and (**??**) split for all $k, n \in \mathbb{N}$.*

Let us recall that the short exact sequences (**??**) and (**??**) also exist if Σ is a non-orientable surface (and similar results hold, [**?**, **?**]) . If $n \geq 2$ and $k \geq 2$ then the splitting of (**??**) for compact surfaces without boundary (also possibly non-orientable) is an open question. As remarked in [**?**] the splitting of one of the two short exact sequences (**??**) and (**??**) does not imply in general that the other sequence splits.

Let us finish this Section by pointing out another interesting feature when we pass from short exact sequences of classical braid groups to short exact sequences of surface braid groups. The sequence (PB) (in the case $k = 1$) has the property that the induced action on the Abelianisation of the kernel is trivial. Following [**?**], we will call such a splitting extension an *almost-direct* product. According to Theorem 3.1 of [**?**] such an exact sequences induces exact sequences on the level of the lower central series quotients (see also [**?**, **?**]). Since P_n is an iterated almost-direct product of free groups, P_n 'inherits' various properties of F_n, and it is possible to use this structure to derive a presentation for the Lie algebra associated to the lower central series of P_n and to construct a universal finite type invariant for braid groups [**?**]. On the other hand, the fact that P_n acts trivially on the Abelianisation of F_n allows us to compose the Artin representation with the Magnus representation, thus yielding the *Gassner representation* (we refer to [**?**] for the details).

In the case of surface pure braids, however, sequences (**??**) are weaker than the sequence (PB), even for surfaces with boundary:

Theorem 5.4. **([?, ?])** *Let Σ be an orientable surface different from $\mathbb{S}^2$ and $\mathbb{T}^2$. The sequence:*

$$1 \longrightarrow P_k(\Sigma \setminus \{x_1, \ldots, x_n\}) \longrightarrow P_{k+n}(\Sigma) \longrightarrow P_n(\Sigma) \longrightarrow 1$$

defines an almost-direct product structure for $P_{k+n}(\Sigma)$ if and only if $n = 1$.

We proved the "if" part in [?]: the "only if" statement follows from the following proposition ([?]), which shows that the existence of an almost-direct product structure is independent of the choice of section.

Proposition 5.5. **([?])** *Let $1 \longrightarrow K \longrightarrow G \xrightarrow{p} Q \longrightarrow 1$ be a split extension of groups. Let s, s' be sections for p, and suppose that the induced action of Q on K via s on the Abelianisation $K^{Ab} = K/[K,K]$ is trivial. Then the same is true for the section s'.*

In [?] we prove that pure braid groups of the torus and of surfaces with boundary components are residually torsion-free nilpotent. This is achieved by showing that they may be realised as subgroups of the Torelli group of a surface of higher genus, which is known to be residually torsion-free nilpotent (see for instance [?]). The embedding that we proposed does not hold when the surface is without boundary. The case of pure braid groups of closed surfaces has been solved in [?] as an easy Corollary of the fact that these groups can be decomposed as almost-direct products of residually torsion free nilpotent groups (see Theorem **??**). Summarizing results from [?, ?] we have therefore that

Theorem 5.5. **([?, ?])** *Let Σ be an oriented surface (possibly with boundary) different from $\mathbb{S}^2$: the group $P_n(\Sigma)$ is residually torsion-free nilpotent.*

We remark that that residual torsion-free nilpotence implies the residual p-finiteness and the bi-orderability. The group $P_n(\Sigma_g)$ (for $g \geq 1$) was proven to be bi-orderable in [?].

On the other hand the residually p-finiteness of $P_n(\Sigma_g)$ can be obtained as a consequence of a general result from [?] stating that the mapping class group of a punctured oriented surface is virtually residually p-finite.

5.4 Surface framed braids

As we observed in previous Section, contrarily to P_n, pure braid groups of closed surfaces do not have a natural structure of iterated semidirect product of free groups. In [?] we introduce the framed braid groups $FB_n(\Sigma)$ and $FP_n(\Sigma)$ of a surface Σ, which generalise respectively the framed braid groups introduced in [?] and the framed pure braid groups considered in Theorem 5.1 of [?]. These groups turn out also to be related to generalisations of Hilden groups introduced in [?] (see also next Chapter).

As defined in Chapter 1, let $F_n(\Sigma) = \Sigma^n \setminus \Delta$, where Δ is the set of n-tuples $x = (x_1, \ldots, x_n)$ for which $x_i = x_j$ for some $i \neq j$. Let $U\Sigma$ be the unit tangent bundle of Σ and $\pi : U\Sigma \longrightarrow \Sigma$ be the natural projection. We denote by $FF_n(\Sigma)$ the subspace $(\pi^n)^{-1}(F_n(\Sigma))$ of $(U\Sigma)^n$ and fix a unit tangent vector v_i of Σ at p_i such that $FF_n(\Sigma)$ is based at $\underline{v} = (p_i, v_i)_{i=1,\ldots,n}$. The symmetric group $\mathfrak{S}_n$ acts freely on $FF_n(\Sigma)$: we denote $\widehat{FF_n}(\Sigma)$ the quotient space $FF_n(\Sigma)/\mathfrak{S}_n$.

The *pure framed braid group* $FP_n(\Sigma)$ on n strands of Σ is the fundamental group of $FF_n(\Sigma)$. The *framed braid group* on n strands of Σ is the fundamental group of $\widehat{FF_n}(\Sigma)$.

For surfaces of genus greater than 1, with boundary or closed, we give two other definitions of these groups, as subgroups of mapping class groups and as subgroups of braid groups of surfaces and we prove their equivalence (in the case of the torus the proposed definitions are not equivalent and involve different notions of framings).

In [?] we prove that, when the surface is closed and of genus greater than 1, these groups are non trivial central extensions of surface pure braid groups.

Theorem 5.6. **([?])** *Let $\Sigma_{g,b}$ a surface of genus $g \geq 2$ with b boundary components.*
1) If $\Sigma_{g,b}$ has boundary, the framed pure braid group $FP_n(\Sigma_{g,b})$ is isomorphic to $\mathbb{Z}^n \times P_n(\Sigma_{g,b})$.
2) If Σ_g is a closed surface, there is a non-splitting central extension

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow FP_n(\Sigma_g) \xrightarrow{\beta_n} P_n(\Sigma_g) \longrightarrow 1 \tag{5.2}$$

where β_n is the morphism induced by the projection map $FF_n(\Sigma_g) \longrightarrow F_n(\Sigma_g)$ (i.e. β_n consists in forgetting the framing).
3) In the two cases, $FF_n(\Sigma_{g,b})$ is an Eilenberg-Maclane space of type $(FP_n(\Sigma_{g,b}), 1)$.

Previous Theorem allows us to provide a group presentation for $FP_n(\Sigma)$ and therefore for $FB_n(\Sigma)$. We show also that the sequence (SPB) defined in previous Section extends naturally to a sequence on framed braids, that we will call *framed surface pure braid sequence* (denoted by (FSPB)) and which splits even in the case of closed surfaces:

Theorem 5.7. **([?])** *For $g \geq 2$, $b \geq 0$, $n \geq 0$ and $m \geq 0$, one has the following splitting exact sequence :*

$$(FSPB) \qquad 1 \longrightarrow FP_m(\Sigma_{g,b+n}) \longrightarrow FP_{n+m}(\Sigma_{g,b}) \xrightarrow{\alpha_{n,m}} FP_n(\Sigma_{g,b}) \longrightarrow 1$$

where $\alpha_{n,m}$ consists in forgetting the first m strands.

Remark 5.2. *Note that the splitting of (FSPB) sequence consists in cabling the first framed strand while the splitting of (SPB) sequence in the case of surfaces with boundary consists in adding m strands at the infinity. If Σ has boundary, adding m framed strands at the infinity gives another splitting of (FSPB) sequence.*

5.5 Applications

We already mentioned that the short exact sequence (PB) plays a central rôle in the study of Vassiliev invariants of braid groups and of Lie Algebras related to pure braid groups; in a completely different direction, the commutator subgroups of P_n are related to (surface) brunnian braids ([?]) and higher homotopy groups of $\mathbb{S}^2$ ([?]).

In what follows, we will focus on the relevance of such short exact sequences for linear representations of braid groups and their topological generalisations. Let us start with the case $k = 1$ and consider B_n as the mapping class group of $\mathbb{D}_n$. We thus obtain an action of B_n on $\mathbb{D}_n$ that induces an action on $\pi_1(\mathbb{D}_n)$: we recover this way the Artin representation of the braid group B_n as a subgroup of the group of automorphisms of F_n (Lemma 1.1 and Section 1.2). Analogously, we have an action of P_n as the pure mapping class group of $\mathbb{D}_n$ on $P_1(\mathbb{D}_n) \simeq F_n$ what is faithful and coincides with the action by conjugation of P_n on $P_1(\mathbb{D}_n)$ defined by the natural section of (PB) sequence defined in Lemma 1.1. Composing the Artin representation with the Magnus representation associated to the length function $p_1 : B_1(\mathbb{D}_n) \longrightarrow \mathbb{Z}$ (see for instance [?]) we obtain the (non reduced) Burau representation of B_n. In the case of the pure braid group, with a similar construction we obtain the Gassner representation of P_n ([?]).

The Burau representation also has a homological interpretation (see for instance Chapter 3 of [?]). In fact consider as before the length function $p_1 : B_1(\mathbb{D}_n) \longrightarrow G_1 = \mathbb{Z} = \langle t \rangle$. Since the action of B_n on $B_1(\mathbb{D}_n)$ commutes with $p_1 : B_1(\mathbb{D}_n) \longrightarrow \mathbb{Z}$, B_n acts on the regular covering $\widetilde{\mathbb{D}}_n$ of $\mathbb{D}_n$.

The induced action on the first homology group of $\widetilde{\mathbb{D}}_n$ is the (reduced) Burau representation of B_n.

Moreover, also for $k > 1$ we may observe that B_n, regarded as the mapping class group of $\mathbb{D}_n$, acts on $\mathbb{F}_k(\mathbb{D}_n)/\mathfrak{S}_k$ and therefore on its fundamental group, $B_k(\mathbb{D}_n)$. The induced action of B_n on $B_k(\mathbb{D}_n)$ coincides with the action by conjugation of B_n on $B_k(\mathbb{D}_n)$ defined by the natural section of (MB). In order to look for (linear) representations, we consider regular coverings associated with normal subgroups of $B_k(\mathbb{D}_n)$, and we try to see if the induced action on homology is well defined. In other words, we wish to study surjections of $B_k(\mathbb{D}_n)$ onto a group G_k subject to certain constraints.

For $k > 1$, let G_k be the group $\mathbb{Z}^2 = \langle q, t \mid [q, t] = 1\rangle$. The corresponding morphism $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$ for $k > 1$ sends the classical braid generators $\sigma_1, \ldots, \sigma_k$ to q and the generators $\zeta_1, \ldots, \zeta_n$, corresponding to the generators of $\pi_1(\mathbb{D}_n)$, to t. Since the action of B_n on $B_k(\mathbb{D}_n)$ commutes with $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$, it turns out that B_n acts on the regular covering of $\mathbb{F}_k(\mathbb{D}_n)/\mathfrak{S}_k$, and the induced action on the Borel-Moore middle homology group of such a covering space is in fact the kth Bigelow-Krammer-Lawrence representation of B_n. In this way, for $k > 1$ we obtain faithful linear representations of B_n (see [**?**] for $k = 2$ and [**?**] for $k > 2$). The faithful linear representations of Bigelow-Krammer-Lawrence of the Artin braid groups B_n (we refer to [**?**] for a complete description) are probably one of the most important recent discoveries in the theory of braid groups, and as such, have been intensively studied over the last few years. They have also been extended to other groups, such as Artin-Tits groups of spherical type (see for instance [**?**, **?**]). However, with the exception of a few results [**?**, **?**, **?**], their generalisation in a more topological direction, to braid and mapping class groups of surfaces for example, as well as the linearity of these groups, are open problems in general.

Let us motivate the choice of the above projections $p_k : B_k(\mathbb{D}_n) \longrightarrow G_k$ using the lower central series of the corresponding groups. If we wish to consider surjections of $B_k(\mathbb{D}_n)$ onto some group G'_k, in order to obtain a linear representation using the approach described above, the group G'_k should be Abelian. Considering the short exact sequence (MB) on the level of Abelianisation, we obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & B_k(\mathbb{D}_n) \arrow[r] \arrow[d, "\bar{q}_k"] & B_{k,n} \arrow[r] \arrow[d, "r_{k,n}"] & B_n \arrow[r] \arrow[d, "r_n"] & 1 \\
1 \arrow[r] & \ker \bar{\psi}_k \arrow[r] & B_{k,n}/\Gamma_2(B_{k,n}) \arrow[r] & B_n/\Gamma_2(B_n) \arrow[r] & 1
\end{tikzcd} \tag{5.3}$$

where $r_{k,n}$ and r_n are Abelianisation homomorphisms, and $\bar{\psi}_k$ is the homomorphism satisfying $\bar{\psi}_k \circ r_{k,n} = r_n \circ \psi_k$. It is straightforward to show that for all $k \geq 1$, $\ker \bar{\psi}_k$ and $\bar{q}_k$ coincide respectively with the group G_k and the morphism p_k considered in Burau (for $k = 1$) and Bigelow-Krammer-Lawrence representations (for $k \geq 1$).

In [**?**], An and Ko described an extension of the Bigelow-Krammer-Lawrence representations of B_n to braid groups of orientable surfaces of positive genus and with non-empty boundary (let us recall that the inclusion of the punctured disk $\mathbb{D}_n$ into a surface Σ different form the sphere induced an embeddding of B_n into $B_n(\Sigma)$, [**?**]). However, it is not currently known whether these representations are faithful. The representation is based on the regular covering arising from a projection of the n-th braid group of a surface Σ with non-empty boundary onto a specific group G_Σ, constructed in a technical manner in order to satisfy certain homological constraints (Section 3 and Definition 2.2 of [**?**]) and which turns out to be a Heisenberg group; more recently, the above projection has been independently studied by Christian Blanchet to obtain a representation of a large subgroup of the Torelli group of a surface with one boundary component containing the Johnson subgroup [**?**].

In [?] we show that technical construction proposed in [?] of the representations may be described in terms of lower central series and exact sequences of surface braid groups. As we mentioned above, in the case of Artin braid groups, the induced short exact sequence on the Γ_2-level gives rise to elements used in the construction of the Bigelow-Krammer-Lawrence representations. In the case of surface braid groups, the corresponding construction on the same level does not work (Proposition 5.4 of [?]), but if we take this construction a stage further, to the Γ_3-level, we obtain the corresponding objects of the An-Ko representations. More precisely we show how to use the Γ_3-level to extend Bigelow-Krammer-Lawrence representations and we prove that such extensions are unique up to isomorphism (Propositions 5.7 and 5.8 of [?]).

We finish this Section with another possible application of the lower central series of surface braid groups by showing that the standard length function on B_n admits a unique extension to a homomorphism whose source is the braid group of a surface of positive genus with one boundary component and that there is no such extension if the surface is closed and orientable.

Proposition 5.6. **([?])** *Let $k, n \geq 3$. Let H be a group, and let λ: $B_n(\Sigma_{g,1}) \longrightarrow H$ be a surjective homomorphism such that $\lambda(B_n)$ is isomorphic to $\mathbb{Z}$. Then there is an isomorphism $\iota : H \longrightarrow B_n(\Sigma_{g,1})/\Gamma_3(B_n(\Sigma_{g,1}))$ for which $\iota \circ \lambda$ sends every generator of $B_n(\Sigma_{g,1})$ to the corresponding class of $B_n(\Sigma_{g,1})/\Gamma_3(B_n(\Sigma_{g,1}))$.*

Proposition 5.7. **([?])** *Let $n \geq 3$ and let Σ be a closed orientable surface of positive genus. It is not possible to extend the length function $\lambda : B_n \longrightarrow \mathbb{Z}$ to $B_n(\Sigma)$. In other words there is no surjection λ_Σ of $B_n(\Sigma)$ onto a group F such that the restriction of λ_Σ to B_n coincides with λ.*

The cases $k = 1, 2$ are still open, mainly because we do not have a finite group presentation for $B_n(\Sigma_{g,1})/\Gamma_3(B_n(\Sigma_{g,1}))$ and $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ in these cases.

Chapter 6

Hilden braid groups

6.1 Closures of (surface) braids

Alexander and plat closure. We recall briefly that we can associate to any classical braid (seen as a collection of paths) "closing up" the ends of the braid as in Figure **??**. Alexander's theorem ensures that we can obtain any link on S^3 as the closure of a braid and, using Markov Theorem, we can characterize all braids having isotopic closures (see for instance Sections 2 and 3 of [**?**] for an exhausting survey on these Theorems and their consequences).

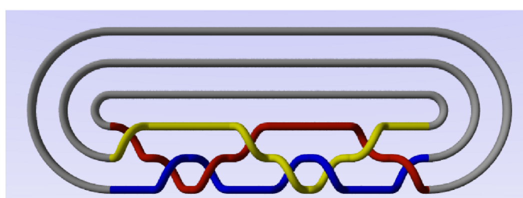


Figure 6.1: Alexander closure.

Another way to connect (usual) braids to links in S^3 is provided by plat closure on braids with an even number of strand. This closure consists by joining consecutive pairs of strings at the top (on the left in Figure **??**) and at the bottom (on the right in Figure **??**).

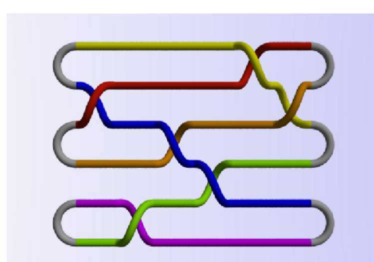


Figure 6.2: Plat closure.

It is evident that any link is isotopic to the plat closure of a braid (Figure **??**) and, as in the case of Alexander closure, one can characterize braids having isotopic plat closure.

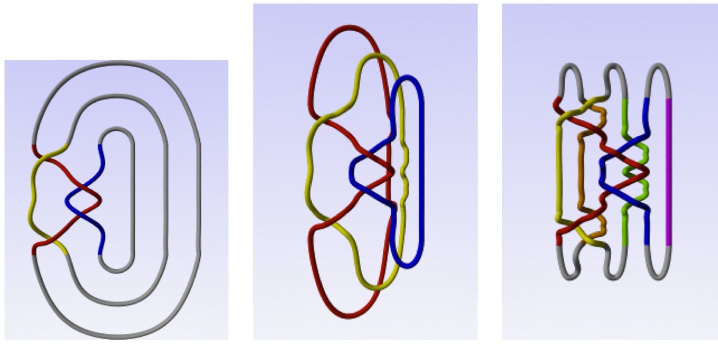


Figure 6.3: From Alexander to plat closure.

In fact, let us denote by $\mathbf{H}_{2n}$ the subgroup of all braids in B_{2n} such that when we join consecutive pairs of strings at the top we get a tangle isotopic to a collection of n trivial arcs. This group, defined in terms of mapping classes (see below) was introduced in [?], motivated by the following characterization of braids having isotopic plat closure.

Theorem 6.1. **([?])** *If two braids $\beta, \gamma \in B_{2n}$ have the same plat closure, then they are related by a sequence of the following moves:*

- *double coset moves, i.e. $\alpha \mapsto h\alpha h'$ for some $h, h' \in H_n$*
- *stabilization moves, i.e. $\alpha \mapsto \tilde{\alpha}\sigma_{2n}$ (or vice-versa), where $\tilde{\alpha}$ is the braid in B_{2n+2} obtained by adding two trivial strands at the bottom of α.*

In [?], Tawn found a finite presentation for $\mathbf{H}_{2n}$. The conjugacy problem in Markov Theorem is replaced by the determinacy problem: given a braid in B_{2n} we can ask if it belongs also to $\mathbf{H}_{2n}$. The determinacy problem for $\mathbf{H}_{2n}$ has been solved in [?]. In a joint work with Tawn we intend to study $\mathbf{H}_{2n}$ as a subgroup of $\mathrm{Aut}(F_{2n})$ and the relations with the welded braid group WB_n.

Surface braids and links in 3 manifolds. It is interesting to remark that Alexander and plat closure give rise to two different notions of closures for surface braids as links in generic 3-manifolds.

In fact, the Alexander closure corresponds to relate surface braids on surfaces with boundary to links in closed 3 manifolds via open book decomposition.

More precisely, let M be a closed 3-manifold and Σ be a compact surface with (non trivial) boundary associated to its open book decomposition: Skora provided in [?] an analogous of Alexander and Markov Theorem stating that any link in M can be seen as the closure (with possibly a non trivial monodromy) of a braid on Σ and giving a characterization of braids having the same closure in M.

On the other hand, in [?] we present a generalization of the notion of plat closure that establishes a relation between braids on closed surfaces and links in 3-manifolds via Heegard splitting.

More precisely, in [?] we introduce a higher genus generalization of $\mathbf{H}_{2n}$, that we called *Hilden braid groups of genus g*. These groups can be seen as subgroups of the ones studied in [?] and they can be thought as a generalization of $\mathbf{H}_{2n}$ in the "braid direction": in Section **??** we show a set of generators for these groups In Section **??**, by fixing a Heegaard decomposition of a given 3-manifold, we explain how it is possible to define a closure for $2n$-string braids on the Heegaard surface, thought of as a generalization of the plat closure of classical braids. In this setting, Hilden braid groups play a role similar to $\mathbf{H}_{2n}$ in the plat closure of classical braids. We end the Chapter with the definition of the Hilden map, relating Hilden braid groups to *motion groups* of links in 3 manifolds.

6.2 Hilden groups and topological generalization

In [**?**] Hilden introduced and found generators for two particular subgroups of the mapping class group of the sphere with $2n$ punctures. Roughly speaking these groups consist of (the isotopy classes of) homeomorphisms of the punctured sphere which admit an extension to the 3-ball fixing n arcs embedded in the 3-ball and bounded by the punctures. The interest in these groups was motivated by the theory of links in $\mathbb{S}^3$ (or in $\mathbb{R}^3$). As the group $\mathbf{H}_{2n}$, these groups can be related to plat closure ([**?**]).

Referring to Figure **??**, let H_g be an oriented handlebody of genus $g \geq 0$ and $\partial H_g = \Sigma_g$. A system of n pairwise disjoint arcs $\mathcal{A}_n = \{A_1, \ldots, A_n\}$ properly embedded in H_g is called *trivial* or *boundary parallel* if there exist n disks (the grey ones in Figure **??**) $D_1, \ldots, D_n$, called *trivializing disks*, embedded in H_g such that $A_i \cap D_i = A_i \cap \partial D_i = A_i$, $\partial D_i - A_i \subset \partial H_g$ and $A_i \cap D_j = \emptyset$, for $i, j = 1, \ldots, n$ and $i \neq j$.

By means of the trivializing disks D_i we can "project" each arc A_i into the arc $a_i = \partial D_i - int(A_i)$ embedded in Σ_g and with the property that $a_i \cap a_j = \emptyset$, if $i \neq j$. We denote with $P_{i,1}, P_{i,2}$ the endpoints of the arc A_i (which clearly coincide with the endpoints of a_i), for $i = 1, \ldots, n$.

Let $\mathrm{MCG}_n(H_g)$ be the group of the isotopy classes of orientation preserving homeomorphisms of H_g fixing the set $A_1 \cup \cdots \cup A_n$.

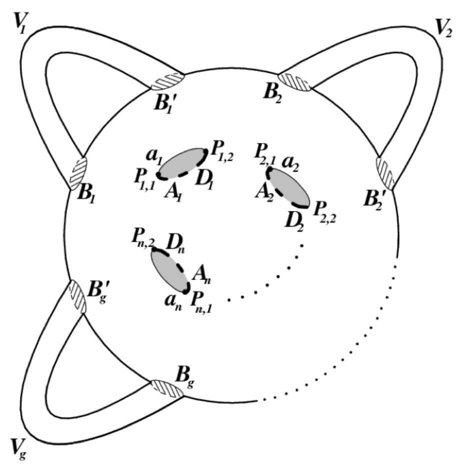


Figure 6.4: The model for a genus g handlebody and a trivial system of arcs.

The *Hilden mapping class group* $\mathcal{E}^g_{2n}$ is the subgroup of the $2n$-punctured mapping class group Γ^{2n}_g defined as the image of the injective group homomorphism $\mathrm{MCG}_n(H_g) \longrightarrow \Gamma^{2n}_g$ induced by restriction to the boundary. In other words, $\mathcal{E}^g_{2n}$ consists of the isotopy classes of homeomorphisms that admit an extension to H_g fixing $A_1 \cup \cdots \cup A_n$. Moreover, we set $\bar{\mathcal{E}}^g_{2n} = \mathrm{P}\Gamma^{2n}_g \cap \mathcal{E}^g_{2n}$ and call it the *pure Hilden mapping class group* (we recall that $\mathrm{P}\Gamma^{2n}_g$ is the subgroups of elements in Γ^{2n}_g fixing $\mathcal{P}_{2n} = \{P_{i,1}, P_{i,2} \,|\, i = 1, \ldots, n\}$ pointwise). As recalled before, the groups $\mathcal{E}^0_{2n}$ and $\bar{\mathcal{E}}^0_{2n}$ were first introduced and studied by Hilden in [**?**].

Now we are ready to define Hilden braid groups. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{MCG}_n(H_g) & \xrightarrow{\cong} & \mathcal{E}^g_{2n} \subset \Gamma^{2n}_g \\
\Big\downarrow {\scriptstyle \bar{\Omega}_{g,n}} & & \Big\downarrow {\scriptstyle \Omega_{g,n}} \\
\mathrm{MCG}(H_g) & \xrightarrow{\cong} & \mathcal{E}^g_0 \subset \Gamma_g.
\end{array}$$

where the vertical rows are forgetful homomorphisms. The *n-th Hilden braid group* of genus g is the group $Hil_n^g := \mathcal{E}_{2n}^g \cap \ker \Omega_{g,n} \cong \ker \bar{\Omega}_{g,n}$. The *n-th Hilden pure braid group* $PHil_n^g$ of genus g is the pure part of Hil_n^g, that is $Hil_n^g \cap \mathrm{P}\Gamma_g^{2n}$. Notice that, since $\Gamma_0 = 1$, then $\mathcal{E}_{2n}^0 = Hil_n^0$ and $\bar{\mathcal{E}}_{2n}^0 = PHil_n^0$. Moreover, as recalled in Chapter 1, $\ker(\Omega_{g,n})$ is isomorphic to the quotient of the braid group $\mathrm{B}_{2n}(\Sigma_g)$ by its center, which is trivial if $g \geq 2$. So, if $g \geq 2$ we can see Hil_n^g as a subgroup of $B_{2n}(\Sigma_g)$.

The definition of $\mathbf{H}_{2n}$ proposed by Tawn [?] is slightly different from ours; indeed, $\mathbf{H}_{2n}$ is a subgroups of B_{2n} instead of the mapping class group Γ_0^{2n}.Nevertheless, from the group presentation of $\mathbf{H}_{2n}$ it is not difficult to obtain a presentation for Hil_n^0 (see [?, ?]).

The main technical result of [?] is to provide a set of generators for Hil_n^g and $PHil_n^g$.

We start by fixing some notations.

Referring to Figure ??, for each $k = 1, \dots, g$, we denote with V_k the k-th 1-handle (i.e. a solid cylinder) obtained by cutting H_g along the two (isotopic) meridian disks B_k and B'_k. Moreover, we set $b_k = \partial B_k$ and $b'_k = \partial B'_k$ and call them meridian curves. For each $i = 1, \dots, n$ the disk D_i denotes the trivializing disk for the i-th arc A_i and $a_i = \partial D_i \setminus \mathrm{int}(A_i)$. The endpoints of both a_i and A_i are denoted with $P_{i,1}, P_{i,2}$ and we set $\mathcal{P}_{2n} = \{P_{i,1}, P_{i,2} \mid i = 1, \dots, n\}$. We fix a disk D embedded in Σ_g containing all the arcs a_i and not intersecting any meridian curve b_k or b'_k. Finally we fix a disk δ_i in D containing a_i and such that $\delta_i \cap a_j = \emptyset$ for $i \neq j$ and $j = 1, \dots, n$.

Let us describe certain families of homeomorphisms of Σ_g fixing setwise $\mathcal{P}_{2n}$ and whose isotopy classes belong to Hil_n^g. We will keep the same notation for a homeomorphism and its isotopy class.

1. **Intervals.** For $i = 1, \dots, n$, we denote with ι_i the homeomorphism of Σ_g that exchanges the endpoints of a_i inside δ_i and that is the identity outside δ_i. We recall that the homeomorphism ι_i is also called braid twist.

2. **Elementary exchanges of arcs.** For $i = 1, \dots, n-1$, let N_i be a tubular neighborhood of $\delta_i \cup \beta_i \cup \delta_{i+1}$ where β_i is a band connecting δ_i and δ_{i+1}, lying inside D and not intersecting any arc a_j, for $j = 1, \dots, n$. We denote with λ_i the homeomorphism of Σ_g that exchanges a_i and a_{i+1}, mapping $P_{i,j}$ to $P_{i+1,j}$ inside N_i, for $j = 1, 2$, and that is the identity outside N_i.

3. **Elementary twists** We denote with s_i the Dehn twist along the curve $d_i = \partial \delta_i$. Notice that $s_i = \iota_i^2$ in Γ_g^{2n}.

4. **Slides of arcs.** Let C be an oriented simple closed curve in $\Sigma_g \setminus \mathcal{P}_{2n}$ intersecting a_i transversally in one point. Consider an embedded closed annulus $A(C)$ in Σ_g whose core is C, containing a_i in its interior part and such that $A(C) \cap \mathcal{P}_{2n} = \{P_{i,1}, P_{i,2}\}$. We denote with C_1 and C_2 the boundary curves of $A(C)$ with the convention that C_1 is the one on the left of C according to its orientation, (see Figure ??).

   The slide $S_{i,C}$ of the arc a_i along the curve C is the multi-twist $T_{C_1}^{-1} T_{C_2} s_i^{\varepsilon}$, where $\varepsilon = 1$ if traveling along C we see $P_{i,1}$ on the right and $\varepsilon = -1$ otherwise. Such an element fixes a_i and determines on Σ_g the same deformation caused by "sliding" the arc a_i along the curve C according to its orientation. We denote the set of all the arc slides by $\mathcal{S}_n^g$.

5. **Admissible slides of meridian disks.** Let $\Sigma_g(i)$ be the genus $g-1$ surface obtained by cutting out from Σ_g the boundary of the i-th handle, and capping the resulting holes with the two meridian disks B_i and B'_i as in Figure ??. A simple closed curve C on $\Sigma_g(i)$ will be called an admissible curve for the meridian disk B_i if it does not intersect $B'_i \cup \mathcal{P}_{2n}$, it intersects B_i in a simple arc and is homotopic to the trivial loop in $\Sigma_g(i) \setminus B'_i$ rel Q, where Q is any point of $B_i \cap C$ (hence homotopies across marked points here are allowed). By exchanging the roles of B_i and B'_i we obtain the definition of admissible curve for the meridian disk B'_i.

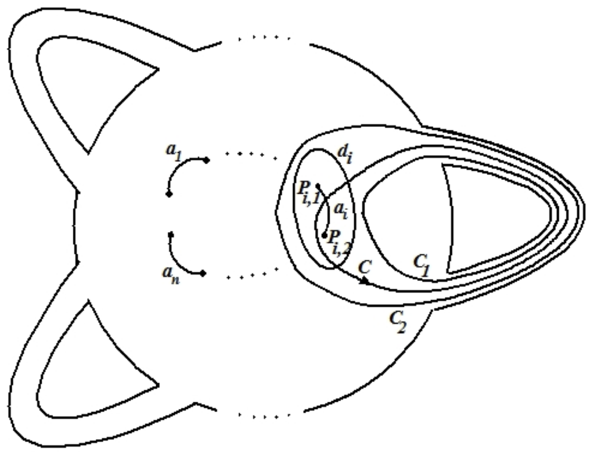


Figure 6.5: The slide $S_{i,C} = T_{C_1}^{-1} T_{C_2} s_i$ of the arc a_i along the curve C.

Let C be an admissible oriented curve for the meridian disk B_i. Let $A(C)$ be an embedded closed annulus in $\Sigma_g(i) \setminus (B_i' \cup \mathcal{P}_{2n})$ whose core is C and containing B_i in its interior part. We denote with C_1 and C_2 the boundary curves of $A(C)$ with the convention that C_1 is the one on the left of C according to its orientation, (see Figure **??**). Notice that one of C_1 and C_2 is homotopic to b_i in $\Sigma_g(i) \setminus B_i'$, while the other is trivial in $\Sigma_g(i) \setminus B_i'$. An *admissible slide* $M_{i,C}$ of the meridian disk B_i along the curve C is the multi-twist $T_{C_1}^{-1} T_{C_2} T_{b_i}^{\varepsilon}$, where $\varepsilon = 1$ if C_1 is homotopic to b_i and $\varepsilon = -1$ otherwise. Since this homeomorphism fixes both the meridian disks B_i and B_i', it could be extended, via the identity on the boundary of i-th handle, to a homeomorphism of Σ_g, and determines on Σ_g the same deformation caused by "sliding" the disk B_i along the curve C according to its orientation. In an analogous way we define $M'_{i,C}$ an admissible slide of the meridian disk B_i' along an admissible oriented curve C for B_i'. We denote the set of all admissible meridian slides with $\mathcal{M}_n^g$.

Remark 6.1. *In [?] one can find explicit extensions of all above homeomorphisms to the couple $(H_g, \mathcal{A}_n)$, that is they all belong to $\mathcal{E}_{2n}^g$. Moreover it is straightforward to see that all the above elements belong also to the kernel of $\Omega_{g,n}$ and so to Hil_n^g.*

It is possible to define the slide of a meridian disk B_i (resp. B_i') along a generic simple closed curve on $\Sigma_g(i) \setminus B_i' \cup \mathcal{P}_{2n}$ (resp. $\Sigma_g(i) \setminus B_i \cup \mathcal{P}_{2n}$). Such a meridian slide still belongs to $\mathcal{E}_{2n}^g$; however, it is easy to see that a slide of a meridian disk belong to the kernel of $\Omega_{g,n}$ (and so to Hil_n^g) if and only if the sliding curve is admissible.

Let $(Z_2)^n \rtimes \mathfrak{S}_n$ be the signed permutation group and let $p : \Gamma_g^{2n} \longrightarrow \mathfrak{S}_{2n}$ be the map which associates to any element of Γ_g^{2n} the permutation induced on the punctures. The next proposition shows that Hil_n^g is generated by ι_1, λ_i, for $i = 1, \dots, n-1$ and a set of generators for $PHil_n^g$.

Proposition 6.1. *The exact sequence*

$$1 \longrightarrow P\Gamma_g^{2n} \longrightarrow \Gamma_g^{2n} \longrightarrow \mathfrak{S}_{2n} \longrightarrow 1$$

restricts to an exact sequence

$$1 \longrightarrow PHil_n^g \longrightarrow Hil_n^g \longrightarrow (Z_2)^n \rtimes \mathfrak{S}_n \longrightarrow 1.$$

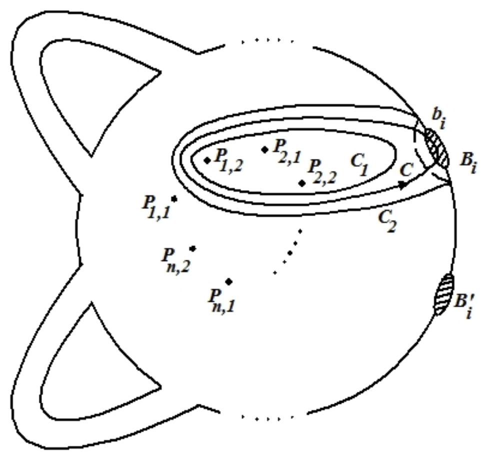


Figure 6.6: The slide $M_{i,C} = T_{C_1}^{-1} T_{C_2} T_{b_i}^{-1}$ of the meridian disk B_i along the curve C.

We say that an element $\sigma \in P\mathrm{Hil}_n^g$ is an *arcs-stabilizer* if σ is the identity on a_i, for each $i = 1, \dots, n$. The set of all arcs-stabilizer elements of $P\mathrm{Hil}_n^g$ determines a subgroup of $P\mathrm{Hil}_n^g$ that we call the *arcs-stabilizer subgroup*.

The following theorem provides an infinite set of generators for $P\mathrm{Hil}_n^g$.

Theorem 6.2. *The group $P\mathrm{Hil}_n^g$ is generated by $\mathcal{M}_n^g \cup \mathcal{S}_n^g \cup \{s_1, \dots, s_n\}$, that is $P\mathrm{Hil}_n^g$ is generated by admissible slides of meridian disks, slides of arcs and elementary twists.*

The proof is quite long and technical: let $\mathcal{G}_n^g$ be the subgroup of $P\mathrm{Hil}_n^g$ generated by $\mathcal{M}_n^g \cup \mathcal{S}_n^g$. The idea is to prove that for any $\sigma \in P\mathrm{Hil}_n^g$ there exists an element $h \in \mathcal{G}_n^g$ such that $h\sigma$ is an arcs-stabilizer. The proof is therefore completed by the following Proposition.

Proposition 6.2. *Let $FP_n(\Sigma_g)$ be the n-th framed braid group of Σ_g defined in Chapter 5. The arcs-stabilizer subgroup of $P\mathrm{Hil}_n^g$ is isomorphic to $FP_n(\Sigma_g)$. In particular the arcs-stabilizer subgroup of $P\mathrm{Hil}_n^g$ is generated by the elementary twists s_i and the slides $m_{i,j}$ and $l_{i,j}$ of the arc a_i along the curves $\mu_{i,j}$ and $\lambda_{i,j}$ depicted in Figure **??**, for $i = 1, \dots, n$ and $j = 1, \dots, g$.*

In order to find a finite set of generators it would be enough to show that the subgroup of $P\mathrm{Hil}_n^g$ generated by $\mathcal{S}_n^g$ and the one generated by the $\mathcal{M}_n^g$ are finitely generated. The next two propositions show that the first subgroup is finitely generated, and the second one is finitely generated when $g = 1$.

Proposition 6.3. *The subgroup of $P\mathrm{Hil}_n(\Sigma_g)$ generated by $\mathcal{S}_n^g$ is finitely generated by*

(1) the elementary twist s_i, for $i = 1, \dots, n$;

*(2) the slides $m_{i,j}$ and $l_{i,j}$ of the arc a_i along the curves $\mu_{i,j}$ and $\lambda_{i,j}$ depicted in Figure **??** for $i = 1, \dots, n$ and $j = 1, \dots, g$;*

*(3) the slides $s_{i,k}$, of the arc a_i along the curve $\sigma_{i,k}$ depicted in Figure **??**, for $1 \le i \ne k \le n$.*

Proposition 6.4. *The subgroup of $P\mathrm{Hil}_n(\Sigma_1)$ generated by $\mathcal{M}_n^1$, the set of admissible meridian slides on the torus, is finitely generated.*

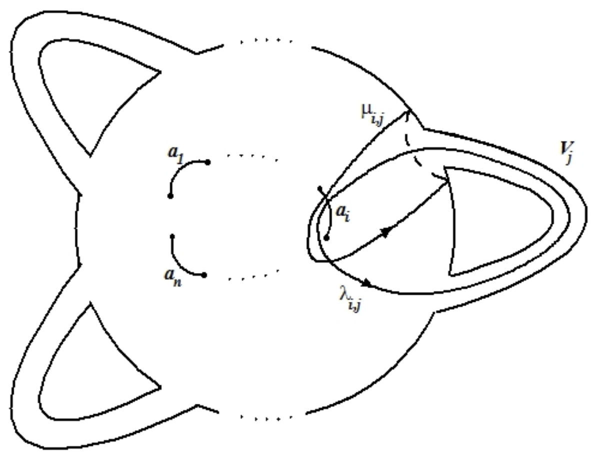


Figure 6.7: The arc slides $m_{i,j}$ and $l_{i,j}$.

From these propositions it follows that $P\mathrm{Hil}_n^1$ is finitely generated. The problem of whether $P\mathrm{Hil}_n^g$ is finitely generated or not when $g > 1$ remains open. We end this section by giving generators for Hil_n^g.

Theorem 6.3. *The group* Hil_n^g *is generated by*

1) ι_1 *and* λ_j, $j = 1, \dots n-1$*;*

2) $m_{1,k}$, $l_{1,k}$, $s_{1,r}$ *with* $k = 1, \dots, g$ *and* $r = 2, \dots n$*;*

3) the elements of $\mathcal{M}_n^g$*;*

that is, Hil_n^g *is generated by elementary exchanges of arcs, the braid twist (interval) of the first arc, the arc slides of the first arc depicted in Figures* **??** *an* **??** *and all admissible meridian slides.*

6.3 Generalized plat closure

In this section we describe a representation of links in 3-manifolds via braids on closed surfaces: this approach generalizes the concept of plat closure and explains the role played by Hil_n^g in this representation. We start by recalling the definition of (g, n)-links.

Let L be a link in a 3-manifold M. We say that L is a (g, n)-*link* if there exists a genus g Heegaard surface S for M such that L intersects S transversally and the intersection of L with both of handlebodies into which M is divided by S, is a trivial system of n arcs.

Such a decomposition for L is called (g, n)-*decomposition* or n-*bridge decomposition of genus* g. The minimum n such that L admits a (g, n)-decomposition is called *genus* g *bridge number* of L.

Clearly if $g = 0$ we get the usual notion of bridge decomposition and bridge number of links in the 3-sphere (or in $\mathbb{R}^3$). Given two links $L \subset M$ and $L' \subset M'$ we say that L and L' are *equivalent* if there exists an orientation preserving homeomorphism $f : M \longrightarrow M'$ such that $f(L) = L'$ and we write $L \cong L'$.

The notion of (g, n)-decompositions was used in [**?**] to develop an algebraic representation of $\mathcal{L}_{g,n}$, the set of equivalence classes of (g, n)-links, as follows. Let $(H_g, \mathcal{A}_n)$ be as in Figure **??** and let $(\bar{H}_g, \bar{\mathcal{A}}_n)$ be a homeomorphic copy of $(H_g, \mathcal{A}_n)$. Fix an orientation reversing homeomorphism

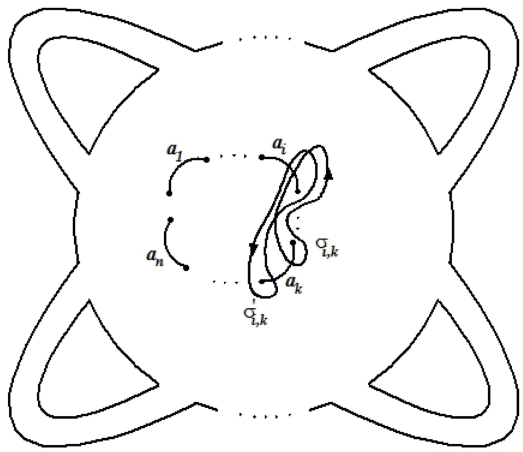


Figure 6.8: The arc slides $s_{i,k}$ and $s'_{i,k}$.

$\tau : H_g \longrightarrow \bar{H}_g$ such that $\tau(A_i) = \bar{A}_i$, for each $i = 1, \ldots, n$. Then the following function is well defined and surjective

$$\Theta_{g,n} : \Gamma_g^{2n} \longrightarrow \mathcal{L}_{g,n} \qquad \Theta_{g,n}(\psi) = L_\psi \tag{6.1}$$

where L_ψ is the (g, n)-link in the 3-manifold M_ψ defined by

$$(M_\psi, L_\psi) = (H_g, \mathcal{A}_n) \cup_{\tau\psi} (\bar{H}_g, \bar{\mathcal{A}}_n).$$

This means that it is possible to describe each link admitting a (g, n)-decomposition in a certain 3-manifold by a element of Γ_g^{2n}. This element is not unique, since we have the following result.

Proposition 6.5. **([?])** *If ψ and ψ' belong to the same double coset of $\mathcal{E}_{2n}^g$ in Γ_g^{2n} then $L_\psi \cong L_{\psi'}$.*

Therefore, in order to describe all (g, n)-links via (**??**) it is enough to consider double coset classes of $\mathcal{E}_{2n}^g$ in Γ_g^{2n}. This representation has been a useful tool for studying links in 3-manifolds, see [**?**, **?**, **?**, **?**]. However, if we represent links using (**??**), we have to deal with links that lie in different manifolds. If we want to fix the ambient manifold, then the following remark holds.

Remark 6.2. *If $\psi_1, \psi_2 \in \Gamma_g^{2n}$ are such that $\Omega_{g,n}(\psi_1) = \Omega_{g,n}(\psi_2)$ then L_{ψ_1} and L_{ψ_2} belong to the same ambient manifold.*

So, in order to fix the ambient manifold, we want to modify representation (**??**) by separating the part that determines the manifold from the part that determines the link.

Referring to Figure **??**, let D be a disk embedded in Σ_g containing all the arcs a_i and not intersecting any meridian curve b_k or b'_k, for $i = 1, \ldots, n$ and $k = 1, \ldots, g$. Let $\mathcal{T}_n^g$ be the subgroup of Γ_g^{2n} generated by Dehn twist along curves that do not intersect the disk D. We have the following proposition.

Proposition 6.6. *For each $\psi \in \mathcal{T}_n^g$ the link L_ψ is an n-component trivial link in M_ψ.*

Let $\Sigma_{g,1}$ be the compact surface obtained by removing the interior part of the disk D. The natural inclusion of $\Sigma_{g,1}$ into Σ_g with $2n$ marked points induces an injective map $\Gamma_{g,1} \longrightarrow \Gamma_g^{2n}$ (see [**?**]) and the mapping class group $\Gamma_{g,1}$ turns out to be isomorphic to the group $\mathcal{T}_n^g$. On the other hand, we have also the following exact sequence:

$$1 \longrightarrow \pi_1(U\Sigma_{g,1}) \longrightarrow \mathcal{T}_n^g \longrightarrow \Gamma_g \longrightarrow 1$$

where $U\Sigma_{g,1}$ is the unit tangent bundle of $\Sigma_{g,1}$ (see Chapter 5). As a consequence each element of Γ_g admits a lifting as an element of $\mathcal{T}_n^g$, so we can realize any genus g Heegaard decomposition of a 3-manifold M using an element of $\mathcal{T}_n^g$, for any $n > 0$. Now we are ready to define the *generalized plat closure*. Let M be a fixed manifold, and choose an element $\psi \in \mathcal{T}_n^g$ such that $M = M_\psi$. We define a map

$$\Theta_{g,n}^{\psi} : \ker(\Omega_{g,n}) \longrightarrow \{(g,n) - \text{links in } M_\psi\} \tag{6.2}$$

given by $\Theta_{g,n}^{\psi}(\sigma) = \Theta_{g,n}(\psi\sigma)$. We set $\hat{\sigma}^\psi = \Theta_{g,n}^{\psi}(\sigma)$.

Remark 6.3. *As recalled before,* $\ker(\Omega_{g,n})$ *is isomorphic to the braid group* $\mathrm{B}_{2n}(\Sigma_g)$*, quotiented by its center (trivial if* $g \geq 2$*). This means that we can think of* $\Theta_{g,n}^{\psi}$ *as a generalization of the notion of plat closure for classical braids, as shown schematically in Figure* **??**. *Indeed, for* $g = 0$ *and* $\psi = \mathrm{id}$ *we obtain the classical plat closure. This is the only representation in the case of the 3-sphere (i. e. with* $g = 0$*), since* $\mathcal{T}_n^0$ *is trivial. On the contrary, the generalized plat closure in a 3-manifold different from* $\mathbb{S}^3$ *depends on the choice of the element* $\psi \in \mathcal{T}_n^g$*, and, topologically, this corresponds to the choice of a Heegaard surface of genus* g *for* M.

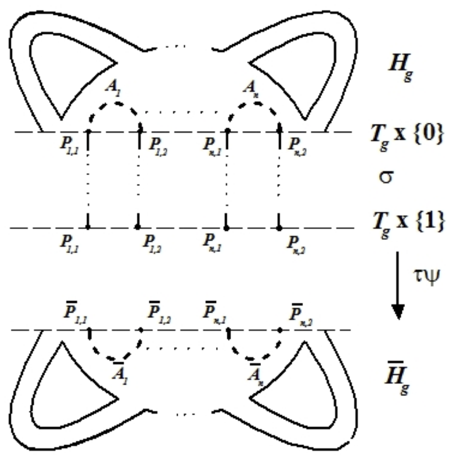


Figure 6.9: A generalized plat closure

In this setting a natural question arises, inspired by Theorem **??**.

Question 6.1. *Is it possible to determine when two element* $\sigma_1 \in \ker(\Omega_{g,n_1})$ *and* $\sigma_2 \in \ker(\Omega_{g,n_2})$ *determine equivalent links via* (**??**)*?*

A partial answer is given by the following statement, which is a straightforward corollary of Proposition **??**.

Corollary 6.1. *Let* $\psi \in \mathcal{T}_n^g$*. Denote with* $\mathrm{Hil}_n^g(\psi) = \psi^{-1}\mathrm{Hil}_n^g\psi$.

1) *if* σ_1 *and* σ_2 *belong to the same left coset of* Hil_n^g *in* $\ker(\Omega_{g,n})$ *then* $\hat{\sigma_1}^\psi$ *and* $\hat{\sigma_2}^\psi$ *are equivalent links in the manifold* M_ψ.

2) *if* σ_1 *and* σ_2 *belong to the same right coset of* $\mathrm{Hil}_n^g(\psi)$ *in* $\ker(\Omega_{g,n})$ *then* $\hat{\sigma_1}^\psi$ *and* $\hat{\sigma_2}^\psi$ *are equivalent links in the manifold* M_ψ.

Another nontrivial question concerns the surjectivity of the map (**??**). The following proposition deals with this problem.

Proposition 6.7. *Let M be a 3-manifold with a finite number of equivalence classes of genus g Heegaard splittings*[1]*. Then there exist $\psi_1, \cdots, \psi_k \in \mathcal{T}_n^g$ such that for each (g,n)-link $L \subset M$ we have $L \cong \hat{\sigma}^{\psi_i}$ with $\sigma \in \ker(\Omega_{g,n})$ and $i \in \{1, \ldots, k\}$.*

The Heegaard genus of a manifold M is the minimal g for which M admits an Heegaard splitting of genus g. For a manifold M if we fix g equal to the Heegaard genus of M, then the previous proposition applies, since in this case the number of equivalence classes of Heegaard splittings is finite (see [?, ?, ?]). In [?] the case $g = b = 1$ is analyzed.

6.4 The Hilden map and the motion groups

In this section we describe the connections between Hilden braid groups and the so-called *motion groups*. We start by recalling few definitions (see [?]).

A *motion* of a compact submanifold N in a manifold M is a path f_t in $Homeo_c(M)$ such that $f_0 = \mathrm{id}$ and $f_1(N) = N$, where $\mathrm{Homeo}_c(M)$ denotes the group of homeomorphisms of M with compact support. A motion is called *stationary* if $f_t(N) = N$ for all $t \in [0, 1]$. The *motion group* $\mathcal{M}(M, N)$ of N in M is the group of equivalence classes of motion of N in M where two motions f_t, g_t are equivalent if $(g^{-1}f)_t$ is homotopic relative to endpoints to a stationary motion.

Notice that the motion group of k points in a surface Σ is the braid group $\mathrm{B}_k(\Sigma)$. Moreover, since each motion is equivalent to a motion that fixes a point $* \in M - N$, when M is non-compact, then it is possible to define a homomorphism

$$\mathcal{M}(M, N) \longrightarrow \mathrm{Aut}(\pi_1(M - N, *)) \tag{6.3}$$

sending an element represented by the motion f_t into the automorphism induced on $\pi_1(M - N, *)$ by f_1. When M is compact, we obtain a homomorphism

$$\mathcal{M}(M, N) \longrightarrow \mathrm{Out}(\pi_1(M - N, *)) \tag{6.4}$$

sending an element represented by the motion f_t into the outer automorphism induced on $\pi_1(M - N, *)$ by f_1.

In [?] a finite set of generators for the motion groups $\mathcal{M}(\mathbb{R}^3, L_n)$ of the n-component trivial link in $\mathbb{S}^3$ is given, while a presentation can be found in [?] (in particular the welded braid group WB_n introduced in Chapter 1 is isomorphic to the subgroup of motions preserving the orientation of any link component). Moreover in [?] a presentation for the motion group of all torus links in $\mathbb{S}^3$ is obtained. On the contrary, there are not known examples of computations of motion groups of links in 3-manifolds different from $\mathbb{S}^3$.

In [?] was explained how to construct examples of motions of a link L in $\mathbb{S}^3$ presented as the plat closure of a braid $\sigma \in \mathrm{B}_{2n}(\mathbb{S}^2)$ using the elements of $\mathrm{Hil}_n^0 \cap \mathrm{Hil}_n^0(\sigma)$. In the following Theorem we extend this result to links in 3-manifolds via Hilden braid groups of a surface.

Theorem 6.4. *Let $\psi \in \mathcal{T}_n^g$ and let $\hat{\sigma}^{\psi}$ be a link in $M_\psi = H_g \cup_{\tau\psi} \bar{H}_g$, where $\sigma \in \ker(\Omega_{g,n})$. There exists a group homomorphism, that we call the Hilden map, $\mathcal{H}_{\psi\sigma} : \mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi\sigma) \longrightarrow \mathcal{M}(M_\psi, \hat{\sigma}^{\psi})$, where $\mathrm{Hil}_n^g(\psi\sigma) = (\psi\sigma)^{-1}\mathrm{Hil}_n^g \psi\sigma$.*

In order to use the Hilden map to get informations on motion groups, it is natural to ask if $\mathcal{H}_{\psi\sigma}$ is surjective and/or injective. Clearly the answer will depend on $\psi\sigma$, that is on the ambient manifold and on the considered link. In [?] we give a (partial) answer in the case of $\mathbb{S}^3$.

[1] Two Heegaard splittings of a manifold M are said equivalent if there exists an homeomorphism $f : M \longrightarrow M$ that send the one splitting surface into the other.

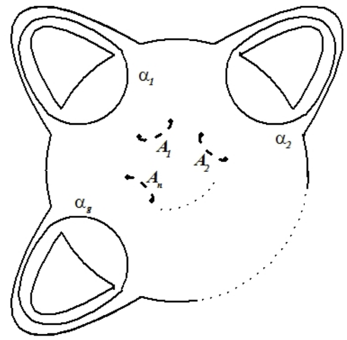


Figure 6.10: An example of Hilden map.

Proposition 6.8. *Let $\psi \in \mathcal{T}_n^g$ be an element such that $M_\psi = \mathbb{S}^3$. For example, choose $\sigma = \mathrm{id}$ if $g = 0$ and $\sigma = T_{\alpha_1} \cdots T_{\alpha_g}$, where $\alpha_1, \ldots \alpha_g$ denote the curves depicted in Figure **??** if $g > 1$. The homomorphism $\mathcal{H}_\psi : \mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi) \longrightarrow \mathcal{M}(\mathbb{S}^3, L_n)$ is surjective, but never injective.*

Using results from [**?**] and [**?**] one could analyze the case of the torus links in $\mathbb{S}^3$. More precisely, once established the surjectivity of the Hilden map it should be interesting to find a geometric representation as well as a group presentation for its kernel: standard methods on exact sequences will therefore provide a group presentation for the source of the Hilden map and then further information on Hilden braid groups themselves. Moreover, the Hilden map could be used in order to get informations on motion groups of links that belong to 3-manifolds different from $\mathbb{S}^3$, in particular a set of generators and, possibly, complete sets of relations for such groups.

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